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Abstract

We have studied some techniques for the synthesis of nonlinear systems. The systems considered here are those that can be characterized by a finite set of Volterra kernels. The approach is to consider the kernels one at a time, by using as basic elements in the synthesis linear systems and multipliers. We present a procedure for testing a given kernel transform to determine whether or not the kernel can be realized exactly with a finite number of linear systems and multipliers. The test is constructive. If it is possible to realize the kernel exactly, a realization is given by the test; if it is not possible to realize the complete kernel exactly, but is possible to break the kernel up into several lower degree components, this will also be discovered by the test. An extension to nonlinear systems of the impulse-train techniques of linear system theory is given. We develop properties of sampling in nonlinear systems, in order to facilitate the use of digital techniques in the synthesis of nonlinear systems. Bandlimiting in nonlinear systems is discussed, and delay-line models for bandlimited systems are given. The transform analysis of nonlinear sampled-data systems by means of the multidimensional z-transform is presented. Computation algorithms for input-output computations are given for direct computation from the multidimensional convolution sum, the associated partial-difference equation, and a decomposition of the nonlinear sampled-data system into linear sampled-data systems. A relationship between time-variant and time-invariant systems is presented, in which time-variant systems are shown to be related to time-invariant systems of higher degree. This enables one to use for linear time-variant systems the properties and techniques developed for second-degree time-invariant systems. A note on the multidimensional formulation of nonlinear systems from the differential equation point of view is given; it is seen that some nonlinear problems in one dimension can be mapped into a linear problem in a higher dimensional space.

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I. INTRODUCTION

1.1 BRIEF HISTORICAL SKETCH

The use of functional analysis as a tool for the study of nonlinear systems was first conceived by the late Norbert Wiener.[1] Following his work, pioneering efforts toward the engineering application of the theory of functional analysis in the representation of nonlinear systems were made by H. E. Singleton,[2] and A. G. Bose,[3] who placed the theory on a firm engineering basis for both discrete and continuous systems. Following a series of lectures at the Massachusetts Institute of Technology by Professor Wiener,[4] several others, among them M. B. Brillant,[5] D. A. George,[6] and D. A. Chesler,[7] studied the theory of continuous nonlinear systems through the use of the Volterra functional power series and the orthogonal Wiener G-functionals. A. D. Hause,[8] George Zames,[9] Martin Schetzen,[10] H. L. Van Trees,[11,12] and D. J. Sakrison[13] subsequently made useful applications and extensions of the theory. Several others have contributed at M.I.T. and elsewhere.

Functional analysis has proved to be a useful tool in the study of a wide class of nonlinear systems. It does not provide an all-encompassing theory; however, particularly when compared with other approaches to the study of nonlinear systems, the startling feature of the theory is not that it does not treat all systems, but rather that the class of systems which it does treat is so very broad.

The basic equation of the theory is

$$y(t) = h_0 + \sum_{n=1}^{\infty} \int \cdots \int h_n(\tau_1, \ldots, \tau_n)\, x(t-\tau_1) \ldots x(t-\tau_n)\, d\tau_1 \ldots d\tau_n, \tag{1}$$

where $x(t)$ is the system input time function, and $y(t)$ is the corresponding output time function. The family of kernels

$$\{h_n(\tau_1, \ldots, \tau_n) : n = 0, 1, 2 \ldots\} \tag{2}$$

characterizes the system, in that knowledge of these kernels provides the means for determining the output corresponding to a given input. Discussion of the scope and properties of (1) and (2) may be found in the work of the authors cited above.

Methods for measuring the kernels of a system have been developed,[14,15] and others are being studied[16]; the correlation methods have been verified experimentally.[17]

It has been observed, originally by Wiener, that the nonlinear system of (1) can be represented as a linear memory section and a nonlinear no-memory section, followed by amplification and summation. This representation provides a basis for a general synthesis procedure for the class of nonlinear systems represented by (1). It is based on the expansion of the input time function in an orthogonal series, with nonlinear no-memory operations being performed on the coefficients of this expansion. Although general and very powerful, it may involve, practically, an unreasonably large amount of equipment.

1.2 THE SYNTHESIS PROBLEM

From both a practical and a theoretical standpoint, it is very desirable to develop means of synthesizing a nonlinear system from the kernels through which it is characterized. The development of some procedures for the synthesis of nonlinear systems is the problem toward which this research is directed.

Several points are inherent in this development of synthesis procedures for nonlinear systems. A finite set of kernels must be adequate for the representation of the system for the inputs of interest. The kernels must be at least approximately realizable with a finite number of components selected for the synthesis. It is to be expected that some means of synthesis may force the abandonment of the use of orthogonal expansions of the input time function and some of the symmetrical properties of the kernels, both extremely valuable properties in the analysis of systems. Also, synthesis procedures as general as that suggested by Wiener should not be expected to be efficient for the same broad class of systems; restrictions on both the inputs to be allowed and the kernels should be expected as the price to be paid in development of synthesis procedures.

Little prior work on the synthesis of nonlinear systems, other than the orthogonal expansion of Wiener, has been done. Jordan[18] found the optimum finite-term orthogonal expansions of the input time function. Van Trees'[11] algorithm for the determination of the optimum compensator for a feedback system provides a solution in terms of the kernels of the optimum system. A thesis at Stanford University by Ming Lei Liou,[19] and work by Schetzen[20] are recent contributions. Schetzen characterized those second- and third-degree kernels that are exactly realizable with a finite number of linear systems and multipliers, while Liou gives a procedure for the recognition of some simple structures of linear systems and polynomial nonlinear no-memory systems.

1.3 THE PRESENT APPROACH

We consider a finite family of kernels

$$\{h_n(\tau_1, \dots, \tau_n) : n = 0, 1, 2, \dots, N\} \tag{3}$$

and attempt to synthesize a system characterized by these kernels. We consider the kernels one at a time and take as elementary building blocks linear systems and multipliers. Any linear system that is realizable in the sense that its unit impulse response, h(t), is zero for $t < 0$ is allowable. After synthesis with these elements is achieved for each kernel of the family, simplification can be attempted, to yield a resulting system that is an interconnection involving linear systems and nonlinear no-memory systems whose input-output characteristic is given by a polynomial. We also consider sampled systems, and the approximation of continuous systems by the sampled systems.

In Section II, the characterization and synthesis of kernels that are exactly realizable with a finite number of linear systems and multipliers is given. A detailed discussion of the effects of sampling in nonlinear systems is presented in Section III. Simulation of continuous systems by sampled systems is discussed in Section IV. In Sections V and VI a multidimensional Z-transform analysis for nonlinear sampled-data systems is developed and used to discuss the synthesis of nonlinear sampled-data systems. An extension to nonlinear systems of the impulse-train techniques which have proved to be so useful for linear systems is discussed in Section VII. In Section VIII we present some miscellaneous results that have been by-products of research into the synthesis problem.

II. KERNELS REALIZABLE EXACTLY WITH A FINITE NUMBER OF LINEAR SYSTEMS AND MULTIPLIERS

Although, as demonstrated by Wiener, it is possible to approximate arbitraril closely any absolutely integrable kernel with a finite number of linear systems and n linear no-memory systems, not all systems representable by a finite set of Volterra kernels can be realized exactly by means of these elements. We consider kernels of the set

$$\{h_n(\tau_1, \ldots, \tau_n) : n = 0, 1, 2, \ldots, N\}$$

one at a time. Since the tests developed to determine whether or not a kernel is exac realizable by means of a finite number of linear systems and multipliers are constru tive tests, we shall not only determine whether or not a kernel is exactly realizable, we shall, if possible, find a realization for the kernel. In the event that some portior of a kernel is exactly realizable but the remainder is not, we shall discover this also, again achieving a realization of as much of the kernel as possible.

We define a kernel transform pair by the relations

$$h_n(\tau_1, \ldots, \tau_n) = \left(\frac{1}{2\pi j}\right)^n \int_{\sigma_n - j\infty}^{\sigma_n + j\infty} \cdots \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} H_n(s_1, \ldots, s_n)\, e^{s_1\tau_1 + \ldots + s_n\tau_n}\, ds_1 \ldots ds_n \tag{4}$$

$$H_n(s_1, \ldots, s_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n(\tau_1, \ldots, \tau_n)\, e^{-s_1\tau_1 - \ldots - s_n\tau_n}\, d\tau_1 \ldots d\tau_n \tag{5}$$

2.1 "CANONIC" OR BASIC FORMS

We shall develop "canonic" or basic forms for kernels that are exactly realiza with a finite number of linear systems and multipliers. These structures are no canonic in any minimal or precise mathematical sense; they are, however, canonic in the sense that any realization by linear systems and multipliers can be pla in these forms.

Consider, first, a second-degree kernel and its transform:

$$h_2(\tau_1, \tau_2) \leftrightarrow H_2(s_1, s_2).$$

It is clear that the most general second-degree system that can be formed with one multiplier is as shown in Fig. 1. The most general second-degree system that can be formed by using N multipliers is shown in Fig. 2. We shall find it convenie to think of the system of Fig. 1 as a canonic form for second-degree systems, sin all second-degree systems that can be realized exactly by means of a finite numbe of linear systems and multipliers can be represented as a sum of these canoni

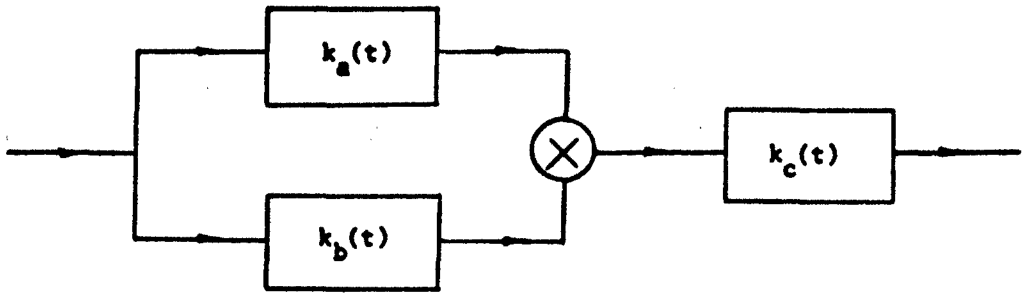


Fig. 1. Canonic second-degree system.

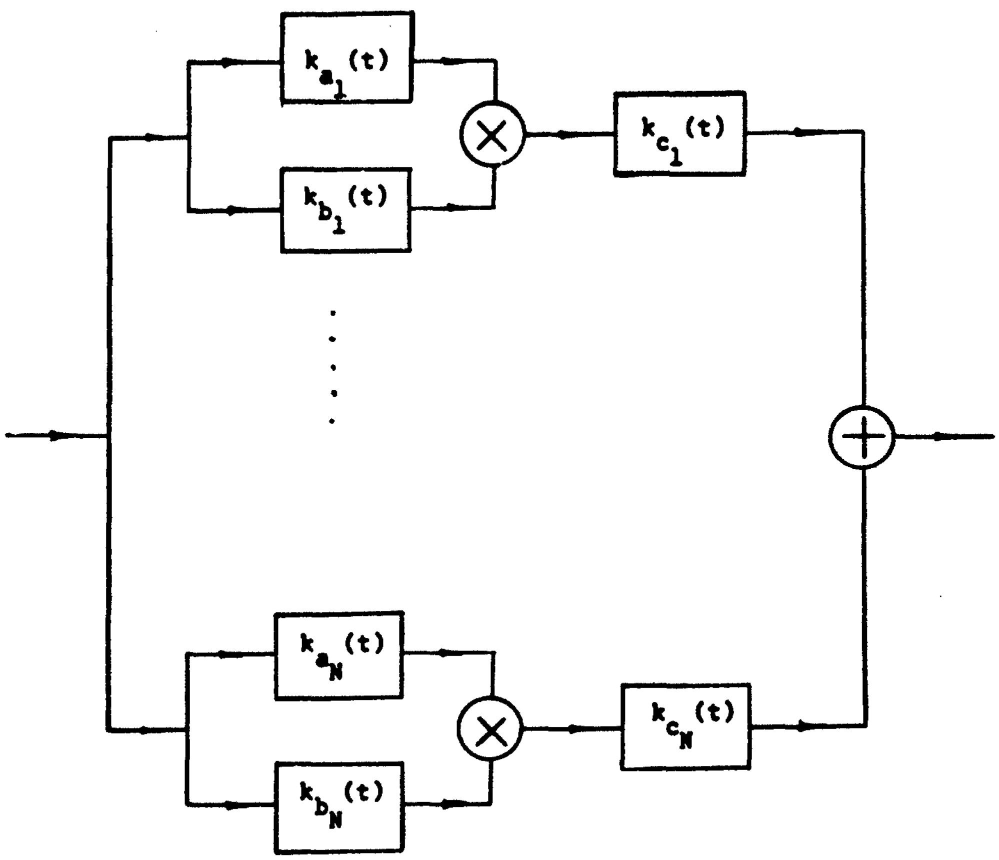


Fig. 2. Second-degree system with N multipliers.

sections, as in Fig. 2.

The kernel of the canonic section of Fig. 1 in terms of the impulse responses $k_a(t)$, $k_b(t)$, and $k_c(t)$ of the linear systems is

$$h_2(\tau_1,\tau_2) = \int k_a(\tau_1-\sigma)\, k_b(\tau_2-\sigma)\, k_c(\sigma)\, d\sigma \tag{6}$$

and the corresponding kernel transform is

$$K_2(s_1,s_2) = K_a(s_1)\, K_b(s_2)\, K_c(s_1+s_2). \tag{7}$$

Then for the system of Fig. 2, we have the kernel and kernel transform

$$g_2(\tau_1,\tau_2) = \sum_{i=1}^{N} \int k_{a_i}(\tau_1-\sigma)\, k_{b_i}(\tau_2-\sigma)\, k_{c_i}(\sigma)\, d\sigma \tag{8}$$

$$G_2(s_1,s_2) = \sum_{i=1}^{N} K_{a_i}(s_1)\, K_{b_i}(s_2)\, K_{c_i}(s_1+s_2). \tag{9}$$

If a given kernel or kernel transform can be expressed in the form (8) or (9), for some N, then it can be realized with at most N multipliers; otherwise it cannot be realized exactly with a finite number of linear systems and multipliers. Examples of both types of systems are given by Schetzen.[20]

Let us now consider higher degree systems. It is clear that the canonic third-degree system is as shown in Fig. 3. It contains five linear systems and two multipliers. In Fig. 4 the same system is shown with the second-degree canonic form composed of $k_a(t)$, $k_b(t)$, and $k_c(t)$ and one of the multipliers shown explicitly. The kernel and the kernel transform of this canonic section are given by

$$k_3(\tau_1,\tau_2,\tau_3) = \iint k_e(\sigma_2)\, k_d(\tau_3-\sigma_2)\, k_c(\sigma_1)\, k_a(\tau_1-\sigma_1-\sigma_2)\, k_b(\tau_2-\sigma_1-\sigma_2)\, d\sigma_1 d\sigma_2 \tag{10}$$

$$K_3(s_1,s_2,s_3) = K_a(s_1)\, K_b(s_2)\, K_c(s_1+s_2)\, K_d(s_3)\, K_e(s_1+s_2+s_3) \tag{11}$$

or by

$$k_3(\tau_1,\tau_2,\tau_3) = \int k_e(\sigma_2)\, k_d(\tau_3-\sigma_2)\, k_2(\tau_1-\sigma_2,\, \tau_2-\sigma_2)\, d\sigma_2 \tag{12}$$

$$K_3(s_1,s_2,s_3) = K_2(s_1 s_2)\, K_d(s_3)\, K_e(s_1+s_2+s_3), \tag{13}$$

where $k_2(\tau_1,\tau_2)$ is the kernel of the second-degree system shown explicitly in Fig. 4.

If a third-degree system has a kernel transform $H_3(s_1,s_2,s_3)$ which can be expressed as

$$H_3(s_1,s_2,s_3) = \sum_{i=1}^{N} K_{a_i}(s_1)\, K_{b_i}(s_2)\, K_{c_i}(s_1+s_2)\, K_{d_i}(s_3)\, K_{e_i}(s_1+s_2+s_3) \tag{14}$$

for some N, then it can be realized exactly with at most 2N multipliers. If it cannot be expressed in this form, then it is impossible to realize the system exactly with a finite number of linear systems and multipliers.

Now for the fourth-degree systems the situation is somewhat more complicated. Consider the systems of Fig. 5 and Fig. 6. Each of them represents a fourth-degree system

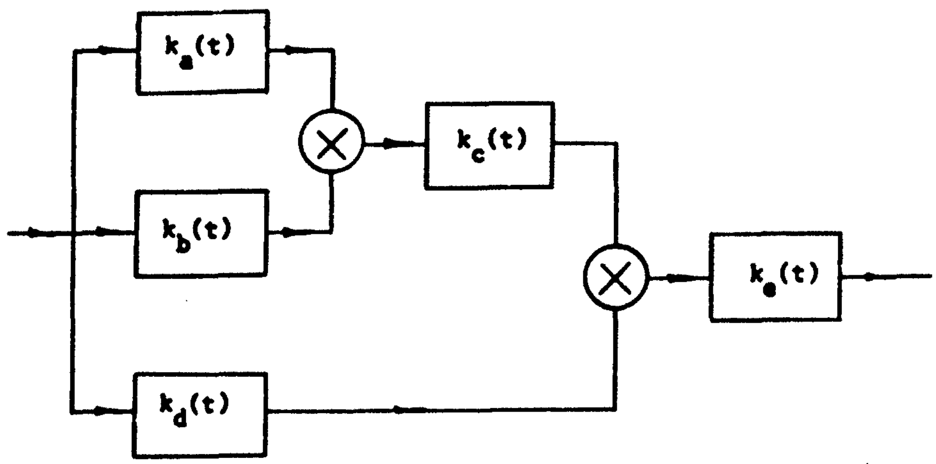


Fig. 3. Canonic third-degree system.

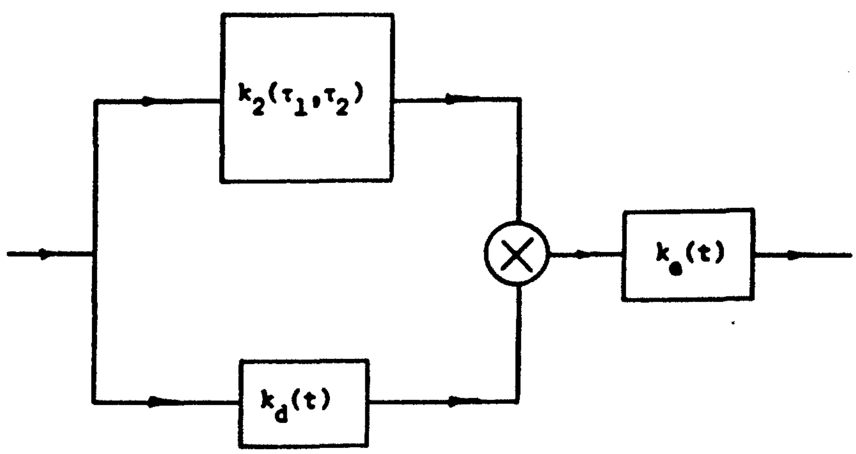
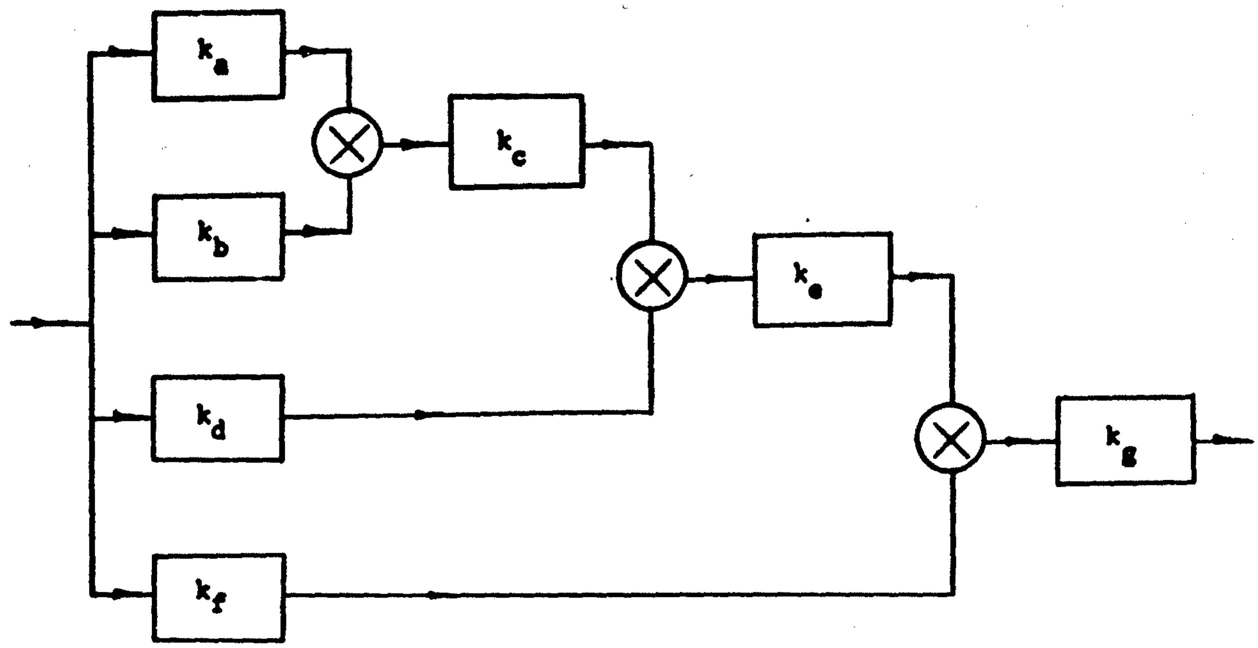
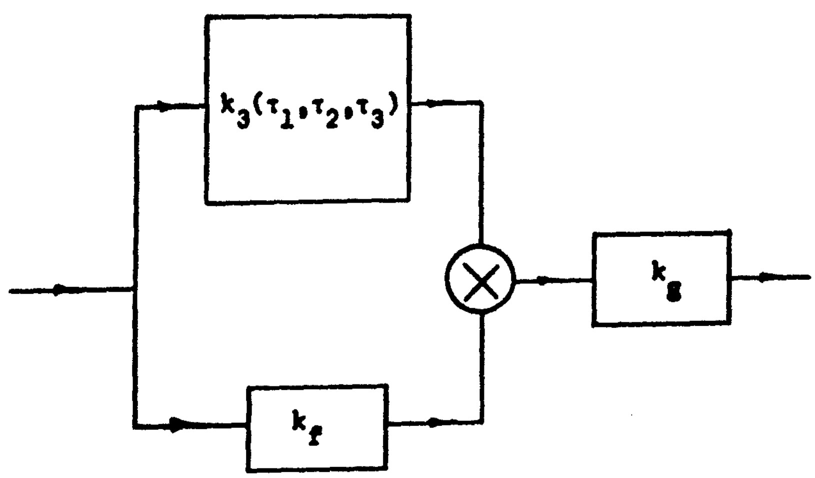


Fig. 4. Alternative form for the canonic third-degree system.

and each of them is composed of seven linear systems and three multipliers, but the two forms are essentially different; that is, no block diagram manipulations can reduce one of these forms to the other. Hence, for fourth-degree systems, we have two canonic sections. It is clear that any fourth-degree system that can be realized with three or



(a)



(b)

Fig. 5. First canonic form for fourth-degree systems.

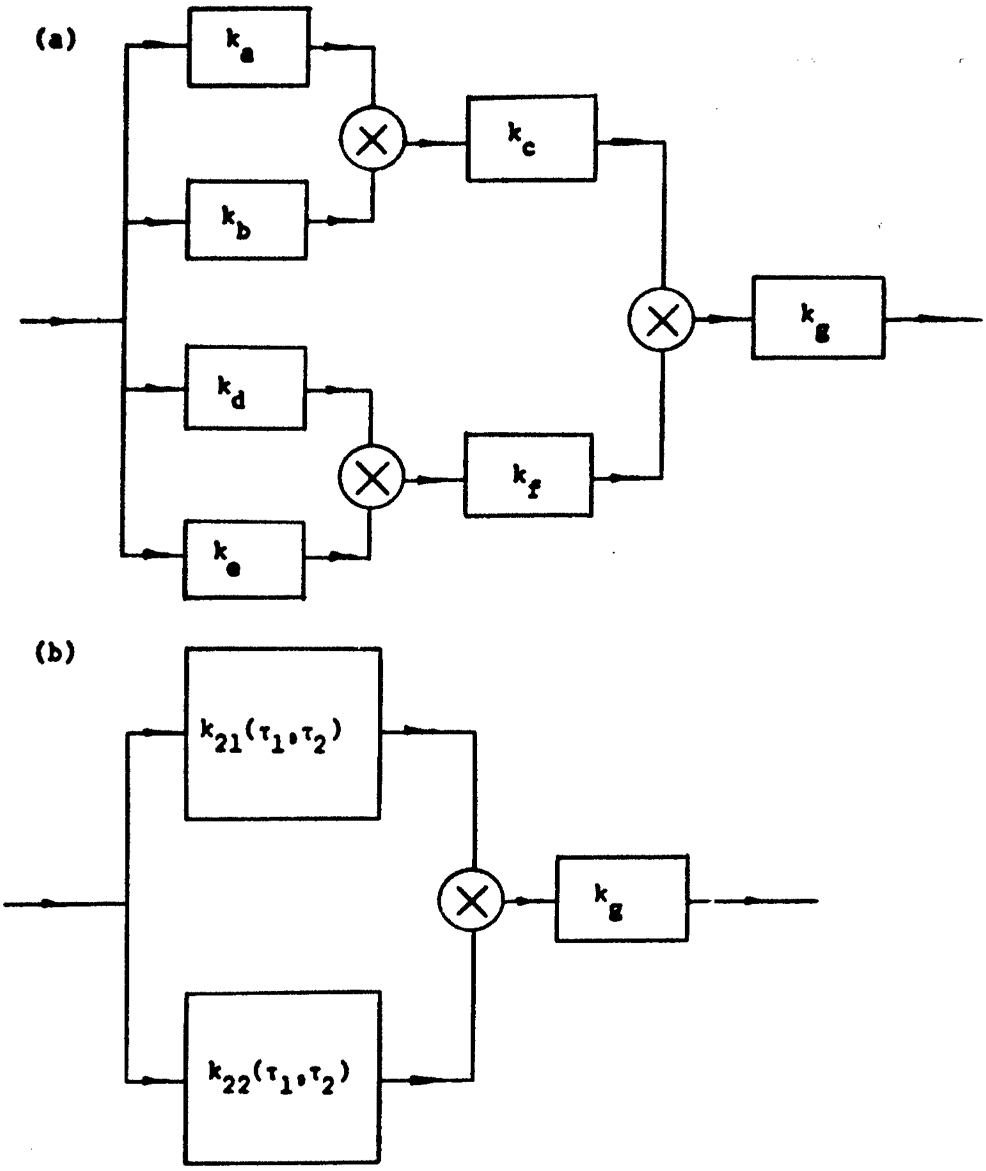


Fig. 6. Second canonic form for fourth-degree systems.

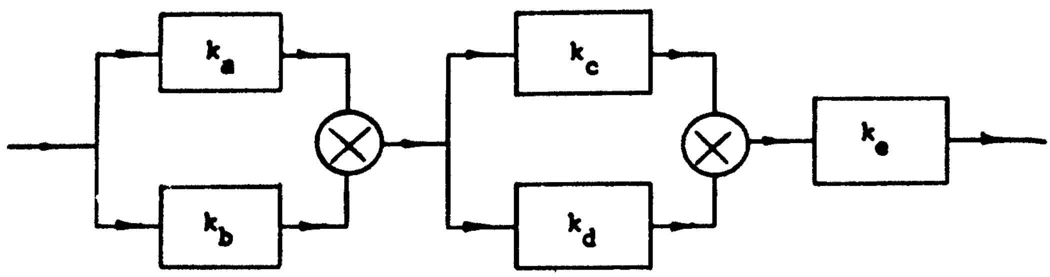


Fig. 7. Fourth-degree system with two multipliers.

less multipliers can be arranged into one of these canonic forms. For example, the fourth-degree system shown in Fig. 7 can be placed in the form of the section of Fig. 6.

The kernel and kernel transforms for the canonic form of Fig. 5 are given by

$$k_{41}(\tau_1,\tau_2,\tau_3,\tau_4) = \iiint k_g(\sigma_3)\, k_f(\tau_4-\sigma_3)\, k_e(\sigma_2)\, k_d(\tau_3-\sigma_2-\sigma_3)\, k_c(\sigma_1)\, k_b(\tau_2-\sigma_1-\sigma_2-\sigma_3)$$
$$k_a(\tau_1-\sigma_1-\sigma_2-\sigma_3)\, d\sigma_1 d\sigma_2 d\sigma_3 \qquad (15)$$

$$K_{41}(s_1,s_2,s_3,s_4) = K_a(s_1)\, K_b(s_2)\, K_c(s_1+s_2)\, K_d(s_3)\, K_e(s_1+s_2+s_3)\, K_f(s_4)\, K_g(s_1+s_2+s_3+s_4) \qquad (16)$$

or

$$k_{41}(\tau_1,\tau_2,\tau_3,\tau_4) = \int k_g(\sigma_3)\, k_f(\tau_4-\sigma_3)\, k_3(\tau_1-\sigma_3,\tau_2-\sigma_3,\tau_3-\sigma_3)\, d\sigma_3 \qquad (17)$$

$$K_{41}(s_1,s_2,s_3,s_4) = K_3(s_1,s_2,s_3)\, K_f(s_4)\, K_g(s_1+s_2+s_3+s_4), \qquad (18)$$

where $k_3(\tau_1,\tau_2,\tau_3)$ is the kernel of the third-degree section within the fourth-degree section.

For the canonic form of Fig. 6, the kernel and kernel transform are given by

$$k_{42}(\tau_1,\tau_2,\tau_3,\tau_4) = \iiint k_g(\sigma_3)\, k_f(\sigma_2)\, k_c(\sigma_1)\, k_a(\tau_1-\sigma_1-\sigma_3)\, k_b(\tau_2-\sigma_1-\sigma_3)\, k_d(\tau_3-\sigma_2-\sigma_3)$$
$$k_e(\tau_4-\sigma_2-\sigma_3)\, d\sigma_1 d\sigma_2 d\sigma_3 \qquad (19)$$

$$K_{42}(s_1,s_2,s_3,s_4) = K_a(s_1)\, K_b(s_2)\, K_c(s_1+s_2)\, K_d(s_3)\, K_e(s_4)\, K_f(s_3+s_4)\, K_g(s_1+s_2+s_3+s_4) \qquad (20)$$

or

$$k_{42}(\tau_1,\tau_2,\tau_3,\tau_4) = \int k_g(\sigma_3)\, k_{21}(\tau_1-\sigma_3,\tau_2-\sigma_3)\, k_{22}(\tau_3-\sigma_3,\tau_4-\sigma_3)\, d\sigma_3 \qquad (21)$$

$$K_{42}(s_1,s_2,s_3,s_4) = K_{21}(s_1,s_2)\, K_{22}(s_3,s_4)\, K_g(s_1+s_2+s_3+s_4), \qquad (22)$$

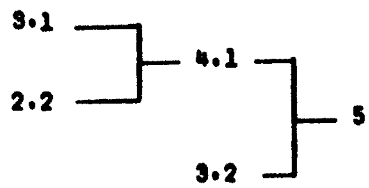
where $k_{21}(\tau_1,\tau_2)$ and $k_{22}(\tau_3,\tau_4)$ represents the second-degree canonic section within the fourth-degree canonic section above.

If a given fourth-degree system is characterized by a kernel transform $H_4(s_1,s_2,s_3,s_4)$ which can be expressed as

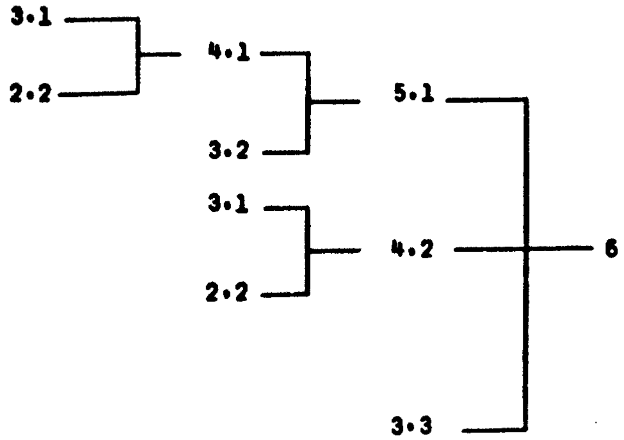
$$H_4(s_1,s_2,s_3,s_4) = \sum_{i=1}^{N_1} K_{41_i}(s_1,s_2,s_3,s_4) + \sum_{i=1}^{N_2} K_{42_i}(s_1,s_2,s_3,s_4) \qquad (23)$$

for some N_1 and N_2, then the system can be realized exactly with at most $3(N_1+N_2)$ multipliers. If the kernel transform cannot be expressed in the form of (23), then it is impossible to realize the system exactly with a finite number of multipliers.

For higher degree systems we shall have more canonic sections. A fifth-degree system may be formed as the product of a fourth-degree and a first-degree system, and the fourth-degree system may be obtained in either of the two forms given above, or we may



(a) Tree for a fifth-degree system



(b) Tree for a sixth-degree system

Fig. 8.
Trees showing canonic structures.

obtain the fifth-degree system as the product of a third-degree system and a second-degree system. A sixth-degree system may be obtained as the product of a fifth-degree system and a first-degree system, a fourth-degree system and a second-degree system, or a third-degree system and a third-degree system, with all possible forms for each.

Although this nomenclature of canonic forms rapidly becomes complex as the degree of the system is increased, we may use the concept of a tree to summarize the process concisely, and to arrive at an expression for the number of different canonic sections existing for an n^{th}-degree system. The trees for the fifth-degree and the sixth-degree cases are shown in Fig. 8.

To form the tree for, say, the fifth-degree system, we proceed from right to left. A fifth-degree system may be formed from the product of a fourth-degree system and a first-degree system, or from the product of a third-degree system and a second-degree system; we need, in moving to the next level of the tree, consider only the fourth-degree, third-degree, and second-degree systems. The third-degree and second-degree systems have only one canonic section; hence the tree branch corresponding to this product terminates there. The fourth-degree system must be broken down further, and hence this branch of the tree continues, spreading out still further, until a level at which only one canonic section exists is reached.

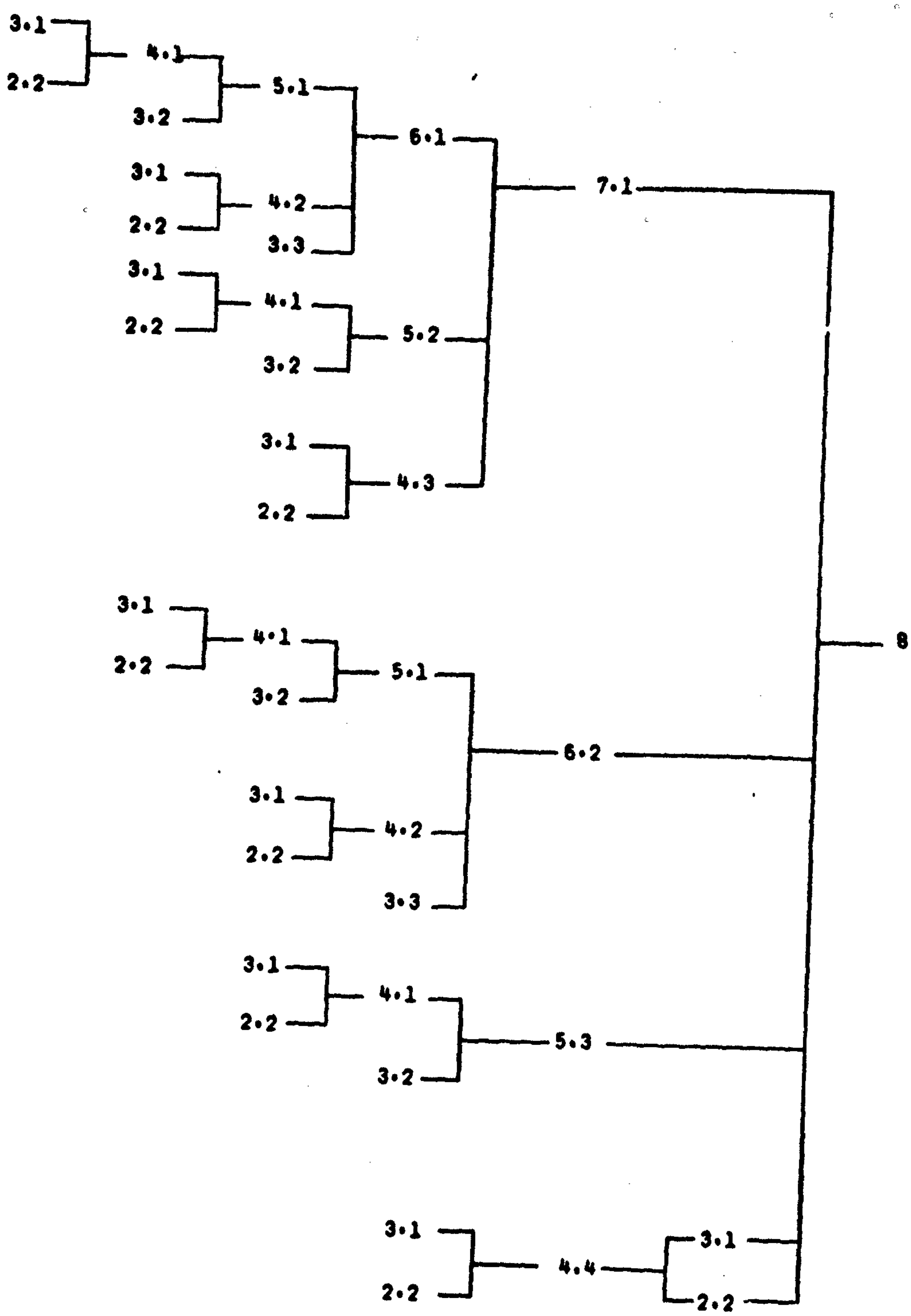


Fig. 9. Tree for an eighth-degree system.

A complication arises in the tree for the eighth-degree case, Fig. 9. Here we find a branch corresponding to a product of two fourth-degree systems; at this point we must expand the tree in both directions until we arrive at branches having only one canonic section, as shown in Fig. 9.

We may now observe that the number of different canonic sections, $C(n)$, for an n^{th}-degree system is obtained from the expression

$$C(n) = \sum_{k=1}^{[n/2]} C(n-k)\, C(k), \qquad n \geq 2$$
$$C(1) = 1, \tag{24}$$

where $[\cdot]$ denotes the greatest integer function. For example, the number of canonic sections for $n = 7$ is given by (24) as

$$C(7) = C(6)\, C(1) + C(5)\, C(2) + C(4)\, C(3) \tag{25}$$

From Fig. 8, or repeated use of (24), we have $C(6) = 6$, $C(5) = 3$, $C(4) = 2$, and $C(3) = C(2) = C(1) = 1$, and hence from Eq. 25, $C(7) = 11$.

For $n = 8$, we have

$$C(8) = C(7)\, C(1) + C(6)\, C(2) + C(5)\, C(3) + C(4)\, C(4) = 24 \tag{26}$$

The tree corresponding to $n = 8$ is shown in Fig. 9.

From the tree we may write the form of the kernel transform of each canonic section by inspection, using the product rule and cascade rule for system combination given by George.[6] For example, from the tree for the fifth-degree system (Fig. 8a) we may write the kernel transforms of each of the three different canonic sections for fifth-degree systems; for the canonic section corresponding to the uppermost path of the tree, the kernel transform is written by inspection as

$$K_5(s_1,s_2,s_2,s_4,s_5) = K_3(s_1,s_2,s_3)\, K_{11}(s_4)\, K_{12}(s_5)\, K_{13}(s_1+s_2+s_3+s_4+s_5), \tag{27}$$

where $K_5(s_1,s_2,s_3,s_4,s_5)$ is the kernel transform of the corresponding fifth-degree canonic section, $K_3(s_1,s_2,s_3)$ is the kernel transform of a third-degree canonic section of (11) and Fig. 3, and $K_{11}(s)$, $K_{12}(s)$, and $K_{13}(s)$ are the kernels of linear systems.

Thus we see that by forming the tree, and following each path in the tree, we may obtain quickly the form in which we must be able to express the kernel transform of an n^{th}-degree system in order that the system be realizable exactly by a finite number of linear systems and multipliers. For higher degree systems the expressions will not be simple, but we have exhibited a procedure for obtaining them with a minimum of effort.

Hence, given the kernel transform of an n^{th}-degree system, we may test that transform to determine whether or not it is realizable exactly with a finite number of linear systems and multipliers.

It should be noted that, in any particular case, it is not necessary to perform the test of a kernel for exact realizability in one step. One proceeds by means of a sequence of simpler tests from the higher degree side of the tree through the lower degree branches as far as possible. Following any path completely through the tree indicates that exact synthesis with linear systems and multipliers is possible; if it is impossible to follow any path completely through the tree such synthesis is not possible. Even when an exact synthesis is not found, proceeding as far as possible through the tree reduces the synthesis problem from the synthesis of one higher degree kernel to the synthesis of several lower degree kernels, which constitutes a significant reduction.

2.2 EXAMPLES

The procedures discussed above are illustrated in the following examples.

Example 1

Consider the kernel transform

$$H_4(s_1,s_2,s_3,s_4) = s_4 \big/ (s_1s_2s_3s_4+2s_1s_2s_3+s_1s_2s_4+s_1s_3s_4+s_2s_3s_4+2s_1s_2+2s_1s_3+2s_2s_3$$
$$+s_1s_4+s_2s_4+s_3s_4+2s_1+2s_2+2s_3+s_4+2). \tag{28}$$

If this kernel is to be realized exactly with a finite number of linear systems and multipliers, we must be able to express it in the form of (23). To determine whether this is possible, we shall first try to express (28) as

$$H_4(s_1,s_2,s_3,s_4) = F(s_1,s_2,s_3)\, G(s_4)\, H(s_1+s_2+s_3+s_4)$$

or by a similar expression with the variables permuted, which corresponds to one of the branches in the tree for a fourth-degree system. We note that it is only necessary to consider the denominator, which we will denote $D(s_1,s_2,s_3,s_4)$, since we could, if necessary, take the numerator one term at a time. We set the variables equal to zero three at a time to obtain

$$D(0,0,0,s_4) = s_4 + 2 \tag{29}$$

$$D(0,0,s_3,0) = 2s_3 + 2 \tag{30}$$

$$D(0,s_2,0,0) = 2s_2 + 2 \tag{31}$$

$$D(s_1,0,0,0) = 2s_1 + 2. \tag{32}$$

Since Eqs. 29-32 have no common factor other than unity, the only possible $H(\cdot)$ is unity. Also, the only possible $G(s_4)$ is, from (29), $G(s_4) = (s_4+2)$. To see if this is indeed a factor, we divide $D(s_1,s_2,s_3,s_4)$ by (s_4+2) to find that

$$D(s_1,s_2,s_3,s_4) = (s_4+2)(s_1s_2s_3+s_1s_2+s_1s_3+s_2s_3+s_1+s_2+s_3+1). \tag{33}$$

Now, with the help of Eqs. 30-32, we recognize the second factor in (33) as $(s_1+1)(s_2+1)(s_3+1)$, and thus have

$$H_4(s_1,s_2,s_3,s_4) = \left(\frac{s_4}{s_4+2}\right)\left(\frac{1}{(s_1+1)(s_2+1)(s_3+1)}\right). \tag{34}$$

Hence this kernel can be synthesized as shown in Fig. 10.

Example 2

Consider the kernel transform given by

$$
\begin{aligned}
H_4(s_1,s_2,s_3,s_4) = (s_1+s_2) \Big/ (&s_1^2 s_2 s_3^2 s_4 + s_1^2 s_2 s_3 s_4^2 + s_1 s_2^2 s_3^2 s_4 + s_1 s_2^2 s_3 s_4^2 \\
&+ 6s_1 s_2 s_3^2 s_4 + 6s_1 s_2 s_3 s_4^2 + 7s_1^2 s_2 s_3 s_4 + 7s_1 s_2^2 s_3 s_4 \\
&+ 2s_1^2 s_2 s_3^2 + s_1^2 s_2 s_4^2 + 2s_1 s_2^2 s_3^2 + s_1 s_2^2 s_4^2 + 2s_1^2 s_3^2 s_4 \\
&+ 2s_1^2 s_3 s_4^2 + s_2^2 s_3^2 s_4 + s_2^2 s_3 s_4^2 + 42s_1 s_2 s_3 s_4 \\
&+ 12s_1 s_2 s_3^2 + 6s_1 s_2 s_4^2 + 10s_1^2 s_2 s_3 + 6s_1^2 s_2 s_4 \\
&+ 10s_1 s_2^2 s_3 + 6s_1 s_2^2 s_4 + 14s_1^2 s_3 s_4 + 7s_2^2 s_3 s_4 \\
&+ 8s_1 s_3^2 s_4 + 8s_1 s_3 s_4^2 + 5s_2 s_3^2 s_4 + 5s_2 s_3 s_4^2 \\
&+ 4s_1^2 s_3^2 + 2s_1^2 s_4^2 + 2s_2^2 s_3^2 + s_2^2 s_4^2 + 60s_1 s_2 s_3 \\
&+ 36s_1 s_2 s_4 + 56s_1 s_3 s_4 + 35s_2 s_3 s_4 + 8s_1^2 s_2 \\
&+ 8s_1 s_2^2 + 20s_1^2 + 12s_1^2 s_4 + 10s_2^2 s_3 + 6s_2^2 s_4 \\
&+ 16s_1 s_3^2 + 8s_1 s_4^2 + 10s_2 s_3^2 + 5s_2 s_4^2 + 6s_3^2 s_4 \\
&+ 6s_3 s_4^2 + 48s_1 s_2 + 80s_1 s_3 + 48s_1 s_4 \\
&+ 50s_2 s_3 + 30s_2 s_4 + 42s_3 s_4 + 16s_1^2 + 8s_2^2 \\
&+ 12s_3^2 + 6s_4^2 + 64s_1 + 40s_2 + 60s_3 + 36s_4 + 48)
\end{aligned}
\tag{35}
$$

Again, we try to put the denominator $D(s_1,s_2,s_3,s_4)$ into a form corresponding to the $3 \cdot 1$ branch of the fourth-degree tree. We find first

$$D(0,0,0,s_4) = 36s_4 + 48 = 12(3s_4+4) \tag{36}$$

$$D(0,0,s_3,0) = 12s_3^2 + 60s_3 + 48 = 12(s_3+4)(s_3+1) \tag{37}$$

$$D(0,s_2,0,0) = 8s_2^2 + 40s_2 + 48 = 8(s_2+3)(s_2+2) \tag{38}$$

$$D(s_1,0,0,0) = 16s_1^2 + 64s_1 + 48 = 16(s_1+3)(s_1+1) \tag{39}$$

Examination of Eqs. 36-39 shows that no common factor other than 4 exists. Hence no

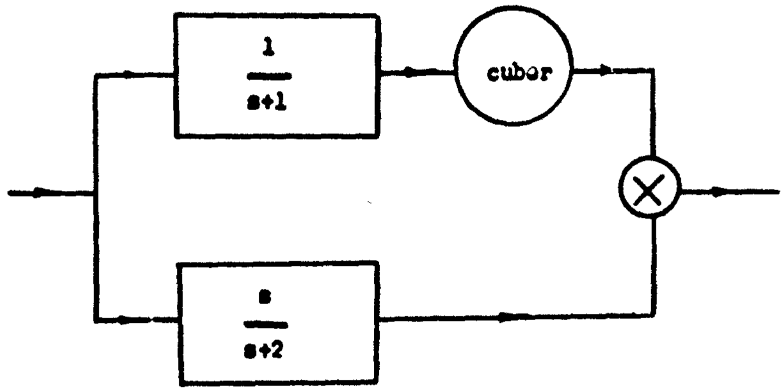


Fig. 10. System with the kernel of Example 1.

factor of the form $H(s_1+s_2+s_3+s_4)$ can exist other than $H(\cdot) = 4$. We next attempt to find a factor that is a function of only one of the variables s_1, s_2, s_3, s_4; however, division of $D(s_1, s_2, s_3, s_4)$ by each of (36)-(39) shows that such a factor cannot exist. We cannot follow the 3·1 branch of the tree.

We then attempt to follow the 2·2 branch, or to express $D(s_1, s_2, s_3, s_4)$ as

$$D(s_1, s_2, s_3, s_4) = H_2(s_1, s_2)\, K_2(s_3, s_4) \tag{40}$$

or by a similar expression with s_1, s_2, s_3, s_4 permuted; we have already found that no nontrivial factor of the form $H(s_1+s_2+s_3+s_4)$ can be present. Hence we write

$$D(0,0,s_3,s_4) = 6(s_3^2s_4+s_3s_4^2+2s_3^2+s_4^2+7s_3s_4+10s_3+6s_4+8). \tag{41}$$

If a factor $K_2(s_3, s_4)$ exists, it must be contained in (41). Division of $D(s_1, s_2, s_3, s_4)$ by (41) is successful, yielding a second factor, and we have reduced (35) to

$$H_4(s_1,s_2,s_3,s_4) = \frac{s_1+s_2}{s_1^2s_2 + s_1s_2^2 + 2s_1^2 + s_2^2 + 6s_1s_2 + 8s_1 + 8s_2 + 6} \left(\frac{1}{s_3^2s_4 + s_3s_4^2 + 2s_3^2 + s_4^2 + 7s_3s_4 + 10s_3 + 6s_4 + 8}\right). \tag{42}$$

We have now reduced the problem to synthesis of two second-degree systems. Note that at this point we are not yet sure whether or not either of these second-degree systems is exactly realizable with a finite number of linear systems and multipliers. We may, however, examine each second-degree system separately, using the techniques of Schetzen,[20] or continuing with the methods used above. We find that we may realize each of them, with the system whose kernel transform is given by (35) then being synthesized as shown in Fig. 11.

Example 3

Consider the second-degree kernel transform given by

$$H_2(s_1,s_2) = \frac{s_1^2\left(1-s_2-e^{-s_2}\right) - s_2^2\left(1-s_1-e^{-s_1}\right)}{s_1^2 s_2^2 (s_2-s_1)}. \tag{43}$$

Denote the numerator of this expression $N(s_1,s_2)$ and the denominator $D(s_1,s_2)$. We observe that the kernel transform cannot be put in the form (9) unless $N(s_1,s_2)$, as well

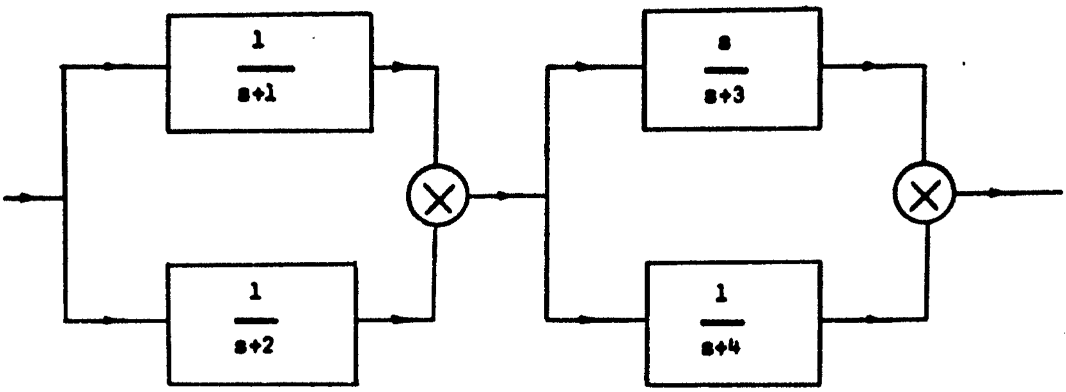


Fig. 11. Realization of the kernel of Example 2.

as $D(s_1,s_2)$, contains the factor (s_2-s_1). Setting $s_1 = s_2 = s$, we find that $N(s,s) = 0$. We next attempt to factor (s_2-s_1) out of $N(s_1,s_2)$. In order to divide out this factor, however, we must expand the exponential terms in their Taylor's series. Before doing this, it is convenient to rewrite $N(s_1,s_2)$ as

$$\begin{aligned} N(s_1,s_2) = {} & s_1^2 - s_1^2 s_2 - s_1^2 e^{-s_2} - s_2^2 + s_1 s_2^2 + s_2^2 e^{-s_1} + s_1 s_2 e^{-s_1} - s_1 s_2 e^{-s_1} \\ & + s_1 s_2 e^{-s_2} - s_1 s_2 e^{-s_2}. \end{aligned} \tag{44}$$

Here we have added and subtracted $s_1 s_2 e^{-s_1}$ and $s_1 s_2 e^{-s_2}$. We may then write $N(s_1,s_2)$ as

$$N(s_1,s_2) = s_1 s_2 (s_2-s_1) - s_2(s_2-s_1)(1-e^{-s_1}) - s_1(s_2-s_1)(1-e^{-s_2}) + s_1 s_2 (e^{-s_1} - e^{-s_2}). \tag{45}$$

We note that (s_2-s_1) is a factor of each of the first three terms; in addition the last term is zero for $s_1 = s_2$, but (s_2-s_1) cannot be factored out to leave a closed-form expression.

We will treat each term separately, writing $H_2(s_1,s_2)$ as

$$H_2(s_1,s_2) = \frac{1}{s_1 s_2} - \frac{1-e^{-s_1}}{s_1^2 s_2} - \frac{1-e^{-s_2}}{s_1 s_2^2} + \frac{e^{-s_1} - e^{-s_2}}{s_1 s_2 (s_2 - s_1)}. \tag{46}$$

We may now focus attention on the last term; of this term, consider the factor, which we denote $F(s_1,s_2)$,

$$F(s_1,s_2) = \frac{e^{-s_1} - e^{-s_2}}{s_2 - s_1}.$$

Expand the exponentials in their Taylor's series, to obtain

$$F(s_1,s_2) = \frac{\left(1 - s_1 + \frac{s_1^2}{2!} - \frac{s_1^3}{3!} + \frac{s_1^4}{4!} - \dots\right) - \left(1 - s_2 + \frac{s_2^2}{2!} - \frac{s_2^3}{3!} + \frac{s_2^4}{4!} - \dots\right)}{s_2 - s_1}$$

$$= \frac{\left(s_2 - s_1 - \frac{s_2^2 - s_1^2}{2!} + \frac{s_2^3 - s_1^3}{3!} - \frac{s_2^4 - s_1^4}{4!} + \dots\right)}{s_2 - s_1}$$

$$= 1 - \frac{s_2 + s_1}{2!} + \frac{s_2^2 + s_1 s_2 + s_1^2}{3!} - \frac{s_2^3 + s_1 s_2^2 + s_1^2 s_2 + s_1^3}{4!} + \dots . \tag{47}$$

Now, although we cannot write this in closed form, and hence cannot realize $H(s_1,s_2)$ exactly with a finite number of linear systems and multipliers, we can approximate $H_2(s_1,s_2)$ with a kernel that is exactly realizable with linear systems and multipliers as closely as we wish by taking as many terms as needed of (47). If we are only interested in the low-frequency behavior of the kernel, only a few terms may suffice. Thus we cannot realize this kernel exactly, but applying the tests we have developed to determine whether or not the kernel was exactly realizable with a finite number of linear systems and multipliers led in this case to an approximation procedure that gives very good approximations for low frequencies.

The following points should be observed from the preceding examples and discussion. First, it is clear that the examination of a kernel to determine if it is exactly realizable may be a lengthy and tedious process. By the use of a tree the labor required is systematized, however. In addition it should be noted that algebraic computation can always be reduced to a first-degree problem, enabling one to use the conventional factorization techniques, although it may be convenient, as it was in Example 3, to do some simplification at a higher level than a one-dimensional problem. Although tedious, the procedure is systematic and feasible, resulting either in a realization of the kernel, the realization of as much of the kernel as possible, or the assurance that nothing can be done to realize the kernel exactly with a finite number of linear systems and multipliers. We may, in the examination of a kernel, find a good approximation even when an exact synthesis is not possible; this is a fortuitous by-product in some cases.

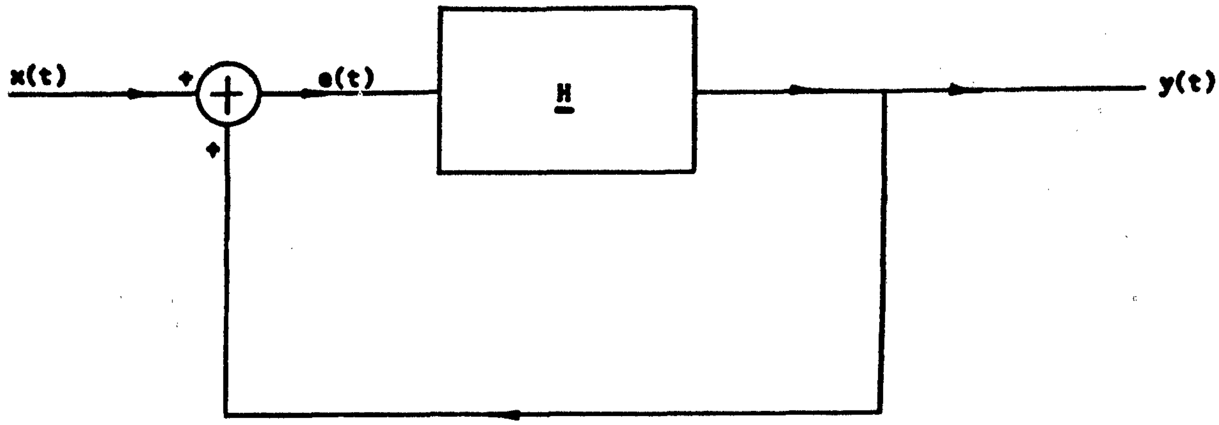


Fig. 12. Nonlinear feedback system.

2.3 COMMENTS ON FEEDBACK STRUCTURES

In Fig. 12 a system characterized by a nonlinear operator $\underline{H}$ is connected in a feedback configuration. The input time function is x(t) and the output time function is y(t); the function on which $\underline{H}$ operates is denoted e(t) = x(t) + y(t). The relation between x(t) and y(t) is given by an operator $\underline{G}$. When nonlinear systems are connected in feedback configurations, as for example in Fig. 12, the input-output relationship of the resulting structure can sometimes be approximated by a finite number of terms of a Volterra series expression; it can never be represented exactly by a finite number of terms. That is, even if the nonlinear operator $\underline{H}$ can be characterized by the second-degree Volterra kernel alone, so that y(t) is a second-degree functional of e(t), it is still not possible to represent y(t) as a finite-term Volterra functional of x(t).

Thus if we consider the kernels of a family one at a time, any exact synthesis must rule out feedback configurations; however, some discussion of feedback structures is pertinent. Zames[21] has tabulated the kernel transforms of the kernels of the Volterra representation of the operator $\underline{G}$ of the feedback structure in terms of the transforms of the Volterra kernels of the nonlinear operator $\underline{H}$. The first few of these is given in Table 1 for convenience. Note that the relationships are formal in nature; if the appropriate Volterra expressions exist, then the relationships hold, but special care must be taken to insure that the Volterra series for the feedback structure actually exists, that is, converges and represents the feedback structure for inputs of interest. We discuss only the formal relationships here, referring to Zames for a discussion of the convergence problem. The expressions in Table 1 differ somewhat from those given by Zames, since we have used positive feedback and Laplace transforms rather than negative feedback and Fourier transforms.

From the expressions of Table 1 we make the following observations. First, suppose that the nonlinear system within the loop, that is, the open-loop relationship, is realizable exactly by a finite number of linear systems and multipliers. Then, we see

Table 1. Kernel transforms for the feedback structure of Fig. 12.

$$G_1(s_1) = \frac{H_1(s_1)}{1-H_1(s_1)}$$

$$G_2(s_1,s_2) = \frac{H_2(s_1,s_2)}{[1-H_1(s_1+s_2)][1-H_1(s_1)][1-H_1(s_2)]}$$

$$G_3(s_1,s_2,s_3) = \frac{1}{1-H_1(s_1+s_2+s_3)} \cdot \frac{1}{\prod_{i=1}^{3}[1-H_1(s_i)]}$$

$$\cdot \left(H_3(s_1,s_2,s_3) + 2\,\frac{H_2(s_1,s_2+s_3)H_2(s_2,s_3)}{1-H_1(s_2+s_3)} \right)$$

that each of the kernels of the closed-loop system is also exactly realizable with a finite number of linear systems and multipliers. Second, given a finite family of kernels, suppose that we have found a realization for each of the kernels in terms of linear systems and multipliers; if the family of kernels can be approximately realized by a feedback structure, then this possibility should be suggested by the repetitive nature of the structure for each of the kernels.

III. SAMPLING IN NONLINEAR SYSTEMS

Digital operations have become recognized as extremely powerful tools in modern control and communication systems. In order to study the possibility of using a digital computer effectively in the study of a nonlinear system we must first understand the effects of sampling in nonlinear systems. We shall now consider nonlinear systems representable by a single term of a Volterra series, and examine in detail the effects of sampling operations at the input and at the output.

3.1 IMPULSES AND NONLINEAR NO-MEMORY OPERATIONS

In the system shown in Fig. 13, the nonlinear system is characterized by the kernel $h_n(\tau_1, \dots, \tau_n)$, $x(t)$ is the input, $x^*(t)$ is the sampled input, $y(t)$ is the output, and $y^*(t)$

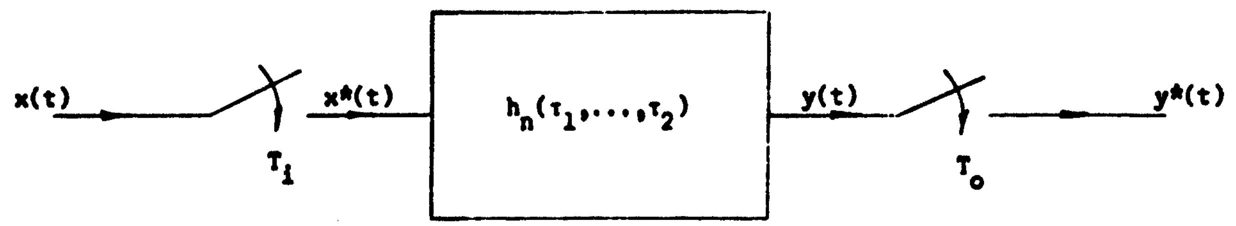


Fig. 13. Nonlinear system with sampled input and output.

is the sampled output. The input and output samplers operate with sampling intervals T_i and T_o, respectively.

The input-output relation for the nonlinear system is

$$y(t) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} h_n(\tau_1, \dots, \tau_n)\, x^*(t-\tau_1) \dots x^*(t-\tau_n)\, d\tau_1 \dots d\tau_n. \tag{48}$$

The kernels $h_n(\tau_1, \dots, \tau_n)$ and the sampled inputs $x^*(t)$ that are permissible deserve close attention. Consider, in this connection, the situation in which a unit impulse is applied to a squarer. Both the squarer and the impulse are ideal models of physical situations. Both are useful models, but together they constitute an incompatible situation.

Regardless of how we represent the impulse as the limiting behavior of a sequence of pulses, and even though we do not look at the limit until we have formed a sequence of output pulses that are the squares of the input pulses, the response of a squarer to an impulse is infinite, not only in amplitude, but also in area.

The difficulty here is in the nature of the models. The squarer places emphasis on the amplitude of its input; no other feature of the input is considered. The impulse, however, places emphasis on the area or weight of a signal. Nonimpulsive inputs and no-memory systems such as squarers are compatible, impulses and linear systems are compatible, but impulses and nonlinear no-memory systems are simply not compatible.

A convenient model for the sampling operation is the "impulse-train modulator." The output of the sampler is taken to be a sequence of impulses at the sampling times, with the areas of the impulses equal to the amplitudes of the input at the sampling instants. Using this model, we have

$$x^*(t) = \sum_{k=-\infty}^{+\infty} x(kT)\, u_o(t-kT). \tag{49}$$

We note that in order for (49) to be meaningful we must exclude impulses from the input $x(t)$; we may permit jump discontinuities in $x(t)$.

But, if this impulsive input $x^*(t)$ is to be presented to a nonlinear system, as in Fig. 13, we must restrict the kernel $h_n(\tau_1, \ldots, \tau_n)$ of the system so that nonlinear no-memory operations on the sampled input are excluded. This is conveniently accomplished by requiring that $h_n(\tau_1, \ldots, \tau_n)$ have no impulsive components.

A common artifice is to require that some type of hold circuit follow the sampler, as in Fig. 14, whenever nonlinear systems are considered. The hold circuit may assume any of several forms,[22] the cardinal hold, zero-order hold, first-order hold, exponential hold, and others.

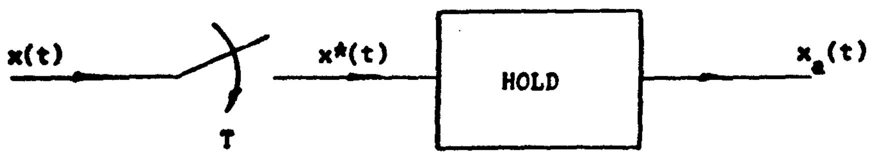


Fig. 14. Sampler with hold.

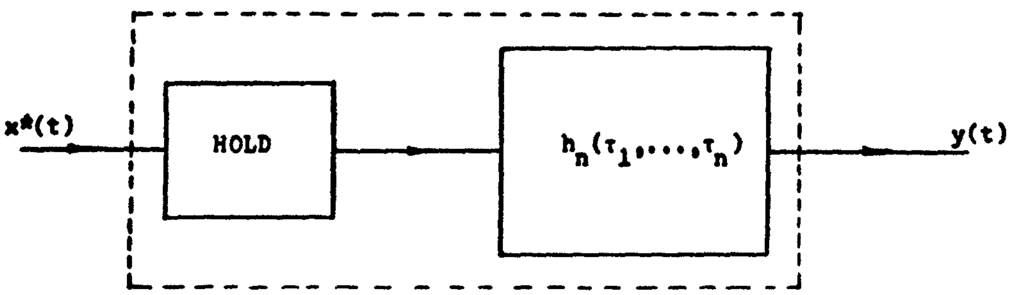


Fig. 15. Nonlinear system with hold.

We may think of the hold circuit in cascade with a nonlinear system, as in Fig. 15, as a new nonlinear system. As long as the hold is linear, the cascade combination may still be represented by an n^{th}-degree kernel; moreover, the kernel of the combination will not contain impulsive components.

When the nonlinear system is characterized by a kernel that has no impulsive

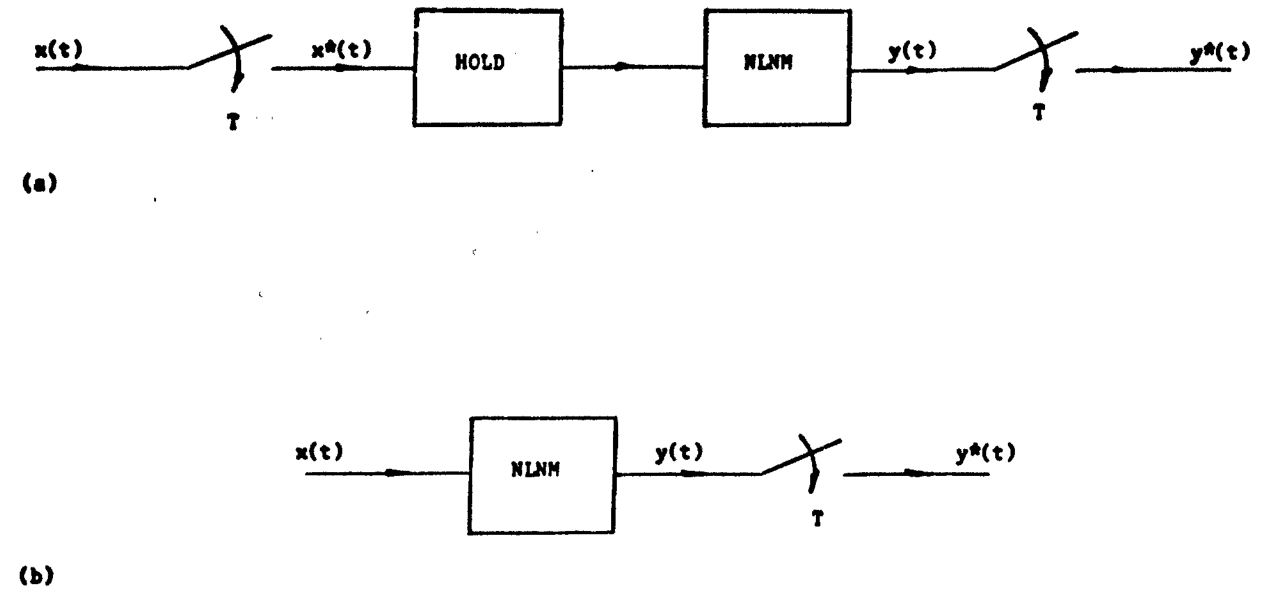


Fig. 16. Models for sampling with nonlinear no-memory systems.

components, that is, when $h_n(\tau_1, \ldots, \tau_n)$ is not impulsive, the hold circuit is not essential to a compatible physical model.

A nonlinear no-memory operation in a situation in which we wish to introduce sampling may be modeled as shown in Fig. 16a or 16b. In either case, $y^*(t)$ is the same, provided that the hold circuit repeats the amplitude of x(t) at the sampling instants; the model of Fig. 16b simply avoids sampling before the nonlinear no-memory operation.

3.2 SECOND-DEGREE SYSTEMS WITH SAMPLED INPUTS

Consider the system shown in Fig. 17. The system N_1 is a second-degree system with the kernel $h_2(\tau_1, \tau_2)$. We exclude kernels having impulsive components. The input x(t) is sampled by an ideal sampler, so that $x^*(t)$ is a sequence of impulses of area x(kT) occurring every T seconds. We assume that s(t) has no impulsive components. The output y(t) is not sampled.

If the output of the sampler is a single-unit impulse, the system output y(t) is given by

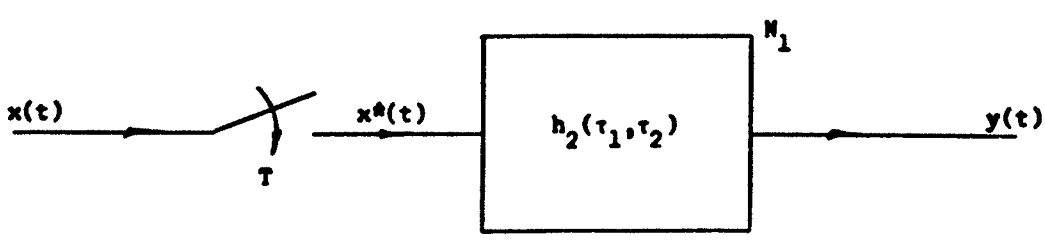


Fig. 17. Second-degree system with sampled input.

$$y(t) = h_2(t, t). \tag{50}$$

Hence the response to a unit impulse is completely determined by the values of the kernel for which the arguments τ_1 and τ_2 are equal; that is, by the values over the line passing through the origin at 45° to each of the axes, as shown in Fig. 18. As indicated,

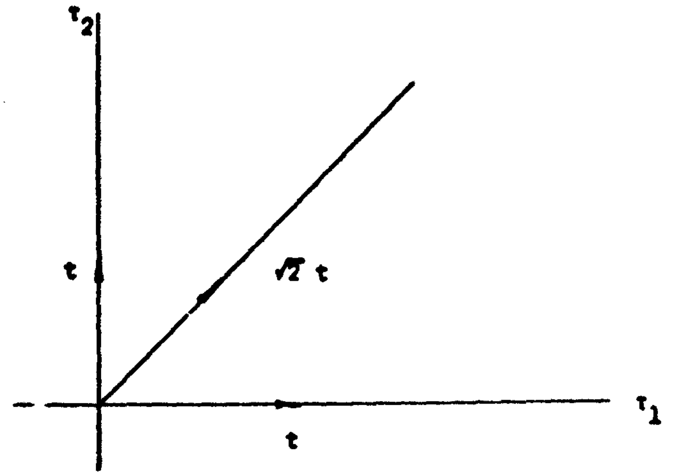


Fig. 18.
Portion of the (τ_1, τ_2)-plane that is significant in the response to a single impulse.

however, the scale along the τ_1 and the τ_2 axes is given by the time scale; hence the scale along the 45° line is $\sqrt{2}$ times the time scale. Thus, although the impulse response is $h_2(t, t)$, if we wish to interpret the section of the surface over the 45° line as the output corresponding to the impulsive input, then we must scale the abscissa values along this line by the factor $1/\sqrt{2}$. This is accomplished conveniently by projecting into either the $\tau_1 = 0$ plane or the $(\tau_2=0)$-plane. To find the response to a unit impulse, then, we look in the $(\tau_1=\tau_2)$-plane and project.

Now suppose that the sampled input is a sequence of unit impulses:

$$x_1^*(t) = \sum_{k=0}^{\infty} u_o(t-kT). \tag{51}$$

Then the corresponding output is found to be

$$y(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} h_2(t-jT, t-kT). \tag{52}$$

Again, the response depends only on certain values of the kernel, and not on the entire surface of the kernel. Observe that in each term of the double sum, the variation in each of the two variables is the same; hence each term is given by the values of the kernel over a line parallel to the $\tau_1 = \tau_2$ line and intersecting the τ_1 or τ_2 axis at some multiple of T units, as indicated in Fig. 19.

Insight can be gained by examining the output y(t) in (52), term by term. The term for $k = j = 0$ is simply $h_2(t, t)$, the response to a unit impulse discussed above. The terms for $k = j = n$ are given by $h_2(t-nT, t-nT)$ and hence are the same as the $k = j = 0$ term, except shifted nT units in time. The situation is clearer if we rewrite (52) as

$$y(t) = \sum_{n=0}^{\infty} h_2(t-nT, t-nT) + \sum_{k>j\geq 0}^{\infty} h_2(t-jT, t-kT) + \sum_{0\leq k<j} h_2(t-jT, t-kT). \tag{53}$$

At $t = 0$, when the first impulse is applied, the output begins just as if the system were a linear system with the unit impulse response $h_2(t, t)$; that is,

$$y(t) = h_2(t, t) \qquad 0 \leq t < T. \tag{54}$$

Now at $t = T$, the second impulse is applied and the output becomes

$$y(t) = h_2(t, t) + h_2(t-T, t-T) + h_2(t, t-T) + h_2(t-T, t), \qquad T \leq t < 2T. \tag{55}$$

If we were dealing with the linear system characterized by $h_2(t, t)$, we would get the first two of these terms. The nonlinear character of the system is now evident; however, we

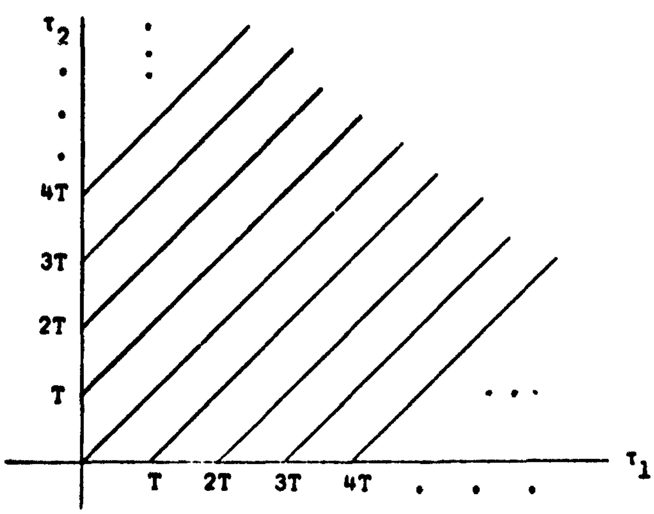


Fig. 19.
Portion of the (τ_1, τ_2)-plane that is significant in the response to a sequence of impulses.

have in addition the last two terms. At time $t = 2T$, all of these terms will start over, just as $h_2(t-T, t-T)$ appeared at $t = T$, and we shall also pick up two more terms, determined by the next two 45° line sections of the kernel of the system. At $t = 3T$ we shall again find new components in the output, and so on.

Eventually, the output will reach a steady-state response. This steady-state response will be the response to an input

$$x_2^*(t) = \sum_{k=-\infty}^{\infty} u_o(t-kT). \tag{56}$$

It is interesting to note that, for the inputs considered, we can think of the output as having been formed by applying the same input to a linear system constructed as follows: Project all the 45° line sections of $h_2(\tau_1, \tau_2)$ into the $\tau_1 = \tau_2$ plane, add all the curves, and project the sum into the $\tau_1 = 0$ plane. The resulting function is the desired impulse response

$$h(t) = h_2(t, t) + \sum_{k=1}^{\infty} [h_2(t-nT, t)+h_2(t, t-nT)]. \tag{57}$$

The response of this linear system and the response of the second-degree system with the kernel $h_2(\tau_1, \tau_2)$ will be identical for inputs (52) and (56). Note, however, that this is a property associated with these specific inputs. If we change the inputs, that is, change the weights of the impulses in the input sequence, the output of the linear system and the output of the second-degree system will no longer be identical. We would need to change the impulse response of (57) to follow the change in the input in order to maintain identical outputs. Thus we abstract from the kernel of the nonlinear system a linear system relating a particular input-output pair.

3.3 HIGHER DEGREE SYSTEMS WITH SAMPLED INPUTS

The response of higher degree systems to sampled inputs retains most of the properties described above for second-degree systems. A single-unit impulse applied to a

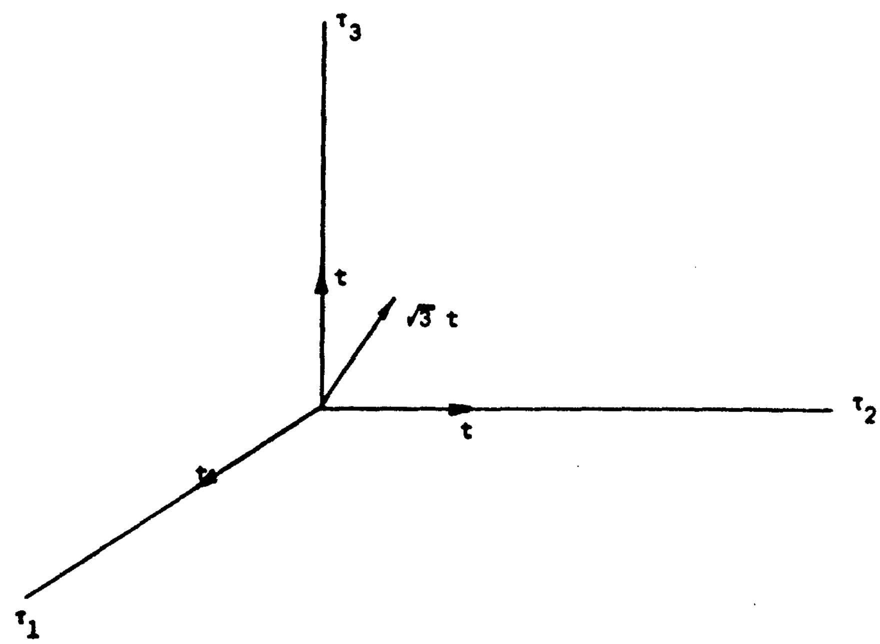


Fig. 20. Portion of τ_1, τ_2, τ_3 space that is significant in determining the response to an impulse.

third-degree system characterized by the kernel $h_3(\tau_1, \tau_2, \tau_3)$ yields an output $h_3(t, t, t)$. In Fig. 20 the portion of the domain of the kernel on which the impulse response depends is shown. The response is completely determined by the values of the kernel for τ_1, τ_2, τ_3 along the line through the origin at 45° to each of the axes. The scale along this line is $\sqrt{3}$ times the time scale along each of the axes.

For an n^{th}-degree system a similar situation applies. The scale factor in the n^{th}-degree case is $\sqrt{n}$, although the graphical interpretation is not possible.

The response of the third-degree system to a sequence of impulses $x_1^*(t)$ is given by

$$y(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} h_3(t-iT, t-jT, t-kT). \tag{58}$$

This sum can be rearranged to be

$$\begin{aligned} y(t) = &\sum_{n=0}^{\infty} h_3(t-nT, t-nT, t-nT) \\ &+ 3 \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} h_3(t-nT, t-nT, t-(n+k)T) \\ &+ 3 \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} h_3(t-nT, t-(n+k)T, t-(n+k)T) \\ &+ 6 \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} h_3(t-nT, t-(n+k)T, t-(n+j+k)T), \end{aligned} \tag{59}$$

where we have grouped together all terms in which all three arguments are equal, in which two of the arguments are equal, and in which no two of the arguments are equal, and have assumed a symmetric kernel.

The response in this case is thus completely determined by the sections of the kernel for which the arguments lie on lines at 45° to each of the axes, intersecting the planes $\tau_1 = 0$, $\tau_2 = 0$, and $\tau_3 = 0$ in a uniformly spaced grid T units on a side, over the positive quadrants of the planes.

For $0 \leq t < T$, the response is $h_3(t, t, t)$; for the next interval we have

$$y(t) = h_3(t, t, t) + h_3(t-T, t-T, t-T) + 3h_3(t, t, t-T) + 3h_3(t, t-T, t-T), \tag{60}$$

and again the nonlinearity becomes evident on application of the second impulse of the sequence. In subsequent intervals more new components will appear in the output. As in the second-degree case, we can abstract from the kernel a linear system relating this specific input-output pair. The impulse response of this linear system is given by

$$h(t) = h_3(t,t,t) + \sum_{k=1}^{\infty} h_3(t,t,t-kT) + 3\sum_{k=1}^{\infty} h_3(t,t-kT,t-kT)$$

$$+ 6\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} h_3(t,t-kT,t-(j+k)T). \tag{61}$$

In the general n^{th}-degree case, the response to $x_1^*(t)$ is

$$y(t) = \sum_{i_n=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} h_n(t-i_1T,\ldots,t-i_nT). \tag{62}$$

Expressions of the form of (61) for the n^{th}-degree case are extremely complicated. It is clear, however, that only certain slices of the kernel are significant in determining the response to a sequence of impulses; for sampled inputs, only the "45° line sections" are important. Thus for sampled inputs, kernels that agree along these lines are completely equivalent.

3.4 INPUT AND OUTPUT SAMPLED

Returning to the situation shown in Fig. 13, consider the relation between the sampled output $y^*(t)$ and the sampled input $x^*(t)$. The sampled output is a sequence of impulses of areas

$$y(pT_o) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} x(k_1T_i)\ldots x(k_nT_i)\, h_n(pT_o-k_1T_i,\ldots,pT_o-k_nT_i), \tag{63}$$

where $x(k_1T_i),\ldots,x(k_nT_i)$ are the input sample values.

We have observed that only the 45° line sections of the kernel surface are important when the input to the system is sampled; from (63) we see that only isolated points on the surface of the kernel are important in determining the sampled output. The remainder of the kernel has no effect on the sampled output.

Sampling at the input restricts attention to the 45° lines; sampling at the output further restricts attention to the points along the 45° lines at intervals of T_o units.

IV. SIMULATION OF CONTINUOUS SYSTEMS BY SAMPLED SYSTEMS

It is often convenient to replace a continuous system by a sampled system; for example, when a digital simulation of a continuous system is to be used. We can take one of two approaches: either we try to find a sampled-data system that performs exactly the same operation as the continuous system or we try to find a sampled-data system that approximates the operation of the continuous system, with the approximation becoming better and better as the sampling interval is made shorter and shorter. The

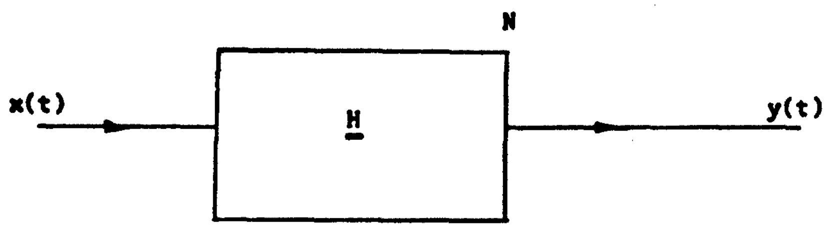


Fig. 21. Nonlinear system.

first approach is useful when the input and the system are bandlimited. If the system is not bandlimited, then we rely on the second approach. We shall discuss the approximation of the convolution integral by a sum approaching the integral as the sampling interval is decreased. Then we shall discuss the bandlimited situation, including the implications of bandlimiting in nonlinear systems, as well as modeling and simulation of these systems.

4.1 APPROXIMATION OF THE CONVOLUTION INTEGRAL BY A SUM

We consider the approximation of a multidimensional convolution integral by a summation approaching the convolution integral in the limit of small sampling intervals. In Fig. 21 a nonlinear system $\underline{H}$ with input x(t) and output y(t) is shown; we shall assume that $\underline{H}$ can be characterized by a single kernel $h_n(\tau_1, \ldots, \tau_n)$ and that $h_n(\tau_1, \ldots, \tau_n)$ and x(t) have no impulsive components. We wish to sample the input and the output and find a sampled-data system that will approximate the continuous system as the sampling interval, T, is decreased.

Consider first the linear case, $n = 1$. The input-output relation is then given by

$$y(t) = \int h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1. \tag{64}$$

There are, of course, many ways to approximate this integral by a sum. We shall discuss a method closely related to and easily modeled by a sampled-data system and subsequently extend this method of approximation and model to higher degree systems.

We construct an approximation to $h_1(\tau_1)$ as follows. Partition the abscissa, τ_1, into uniform intervals of length T, with the origin falling at the center of an interval, as shown in Fig. 22. Construct a stepwise approximation to $h_1(\tau_1)$, taking as the amplitude of each

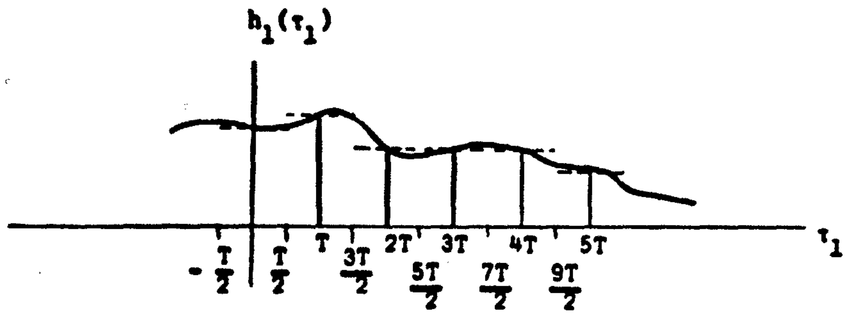


Fig. 22. Approximation of first-degree kernel.

segment the amplitude of $h_1(\tau_1)$ at the midpoint of the interval. Now consider the stepwise approximation to be a sequence of pulses of width T and height $h_1(kT)$. Replace each pulse by an impulse of the same area, occurring at the midpoint of the interval. Then we obtain an approximation

$$h_1(\tau_1) = \sum_{k=-\infty}^{+\infty} h_1(kT)\, u_o(\tau_1 - kT). \tag{65}$$

Substituting (65) in (64) and observing the output y(t) at $t = pT_o$ yields

$$y(pT_o) = \sum_{k=-\infty}^{+\infty} h_1(kT)\, x(pT_o - kT)\, T. \tag{66}$$

As $T \to 0$ and $T_o \to 0$, we make the formal replacements

$$T \to d\tau_1$$
$$kT \to \tau_1$$
$$pT_o \to t$$

and obtain the convolution integral (64). We can represent (66) as shown in Fig. 23 with $n = 1$. The amplifier with gain T is necessary in order that the sampled system approach the continuous system as T is decreased.

For the second-degree case, $n = 2$, the input-output relation of the continuous system is

$$y(t) = \iint h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2. \tag{67}$$

To find an approximating sum for this integral we find an approximation to $h_2(\tau_1, \tau_2)$. Partition both the τ_1 and the τ_2 axes as described above in the linear case; forming the Cartesian product of these partitions yields a uniform partition of the τ_1, τ_2 plane. Form

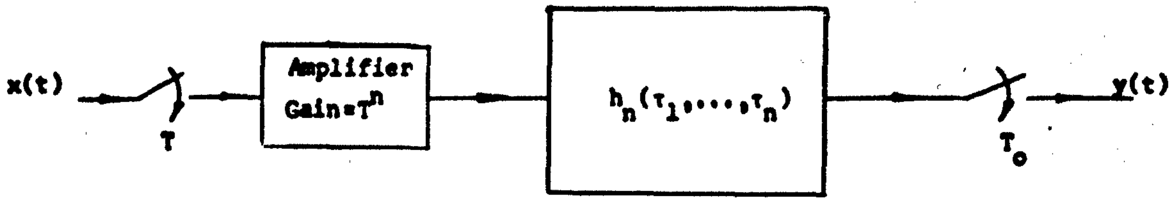


Fig. 23. Approximation of a continuous system.

a stepwise approximation to $h_2(\tau_1,\tau_2)$, taking as the amplitude of each segment the amplitude $h_2(k_1T,k_2T)$ at the midpoint of each two-dimensional interval. Consider this approximation to be a two-dimensional sequence of pulses and replace each pulse with an impulse of the same area occurring at the midpoint to obtain for the two-dimensional kernel

$$h_2(\tau_1,\tau_2) = \sum_{k_2=-\infty}^{+\infty} \sum_{k_1=-\infty}^{+\infty} h_2(k_1T,k_2T)\, u_o(\tau_1-k_1T)\, u_o(\tau_2-k_2T)\, T^2. \tag{68}$$

Substitution of (68) in (67) and observation of the output at $t = pT_o$ yields

$$y(pT_o) = \sum_{k_2=-\infty}^{+\infty} \sum_{k_1=-\infty}^{+\infty} h_2(k_1T,k_2T)\, x(pT_o-k_1T)\, x(pT_o-k_2T)\, T \cdot T. \tag{69}$$

As $T \to 0$ and $T_o \to 0$, we make the formal identifications

$$\begin{array}{ll} T \to d\tau_1 & T \to d\tau_2 \\ k_1T \to \tau_1 & k_2T \to \tau_2 \\ \multicolumn{2}{c}{pT_o \to t} \end{array}$$

to obtain the two-dimensional convolution integral (67). We can represent (69) as shown in Fig. 23 with $n = 2$. The amplifier, in this case having gain T^2, is, as in the linear case, necessary in order to secure the desired limiting behavior.

For the n^{th}-degree case, the approximation above extends readily to give for $h_n(\tau_1,\ldots,\tau_n)$

$$h_n(\tau_1,\ldots,\tau_n) = \sum_{k_n=-\infty}^{+\infty} \cdots \sum_{k_1=-\infty}^{+\infty} h_n(k_1T,\ldots,k_nT)\, u_o(\tau_1-k_1T) \ldots u_o(\tau_n-k_nT)\, T^n \tag{70}$$

and hence for the output $y(pT_o)$ in the n^{th}-degree case

$$y(pT_o) = \sum_{k_n=-\infty}^{+\infty} \cdots \sum_{k_1=-\infty}^{+\infty} h_n(k_1T,\ldots,k_nT)\, x(pT_o-k_1T) \ldots x(pT_o-k_nT)\, T^n. \tag{71}$$

which is represented in block diagram form as a sampled-data system in Fig. 23.

Observing the output at $t = pT_o$, we find that for these summations to have the desired behavior for small T, it is necessary that the stepwise approximation to the kernel be sufficiently accurate, and that the replacement of the pulses by impulses does not introduce too great an error. Also, T_o must be small enough so that we are able to obtain from the samples $y(pT_o)$ a good approximation to $y(t)$.

An alternative development of these results can be obtained by writing (67), or the corresponding equation for linear or higher degree systems, in the equivalent form

$$y(t) = \iint h_2(t-\tau_1, t-\tau_2)\, x(\tau_1)\, x(\tau_2)\, d\tau_1 d\tau_2, \tag{72}$$

which differs from (67) only in a simple change of variables. We may now approximate $x(t)$ in exactly the manner described above for the linear or first-degree kernel, (65) and Fig. 22, to obtain

$$x(t) = \sum_{k=-\infty}^{\infty} Tx(t)\, u_o(t-kT). \tag{73}$$

Substituting (73) in (72) yields (74). In terms of Fig. 23 this amounts to associating the sampler with the input rather than with the system. We arrive at the same type of approximations on the input as described above for the kernel

$$y(pT_o) = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} x(k_1T)\, x(k_2T)\, h_2(pT_o-k_1T, pT_o-k_2T)\, T^2. \tag{74}$$

A change of index in (74) yields (69) again.

From the engineer's point of view, the length of the sampling interval necessary to achieve an adequate approximation constitutes a compromise between conflicting requirements. Shortening the sampling interval requires that more computations be carried out to obtain each sample of the output, and all of the computations will introduce errors unavoidably. The choice of the sampling interval is then a compromise between approximation error and computation error.

4.2 BANDLIMITED SYSTEMS

The concept of bandlimiting is somewhat more complicated in nonlinear than in linear systems; we shall define bandlimiting in linear systems in such a way that the concepts involved will carry over to nonlinear systems.

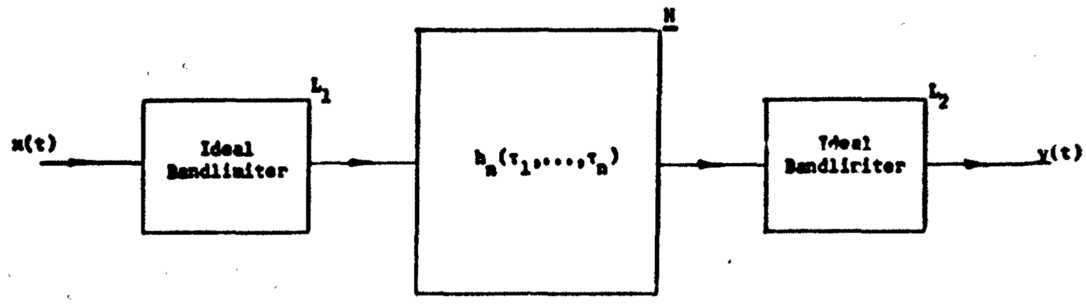


Fig. 24. Nonlinear system with bandlimiting at the input and output.

We say that a linear system is bandlimited if the Fourier transform of the impulse response of the system has a nonzero magnitude only in a certain band or certain bands of frequencies; this definition can be interpreted in terms of properties of the system as observed at the input and the output of the system. Similarly, we may say that a system characterized as an n^{th}-degree kernel is bandlimited if the multidimensional Fourier transform of the kernel of the system has a nonzero magnitude only in a certain region of the multidimensional domain of definition of the kernel. This definition is adequate, but it is not quite as easily interpreted as the corresponding definition for linear systems.

To aid in interpretation of bandlimiting in nonlinear systems, we shall adopt another definition that is equivalent to the one above, both for linear and nonlinear systems, and which has obvious interpretation in both cases.

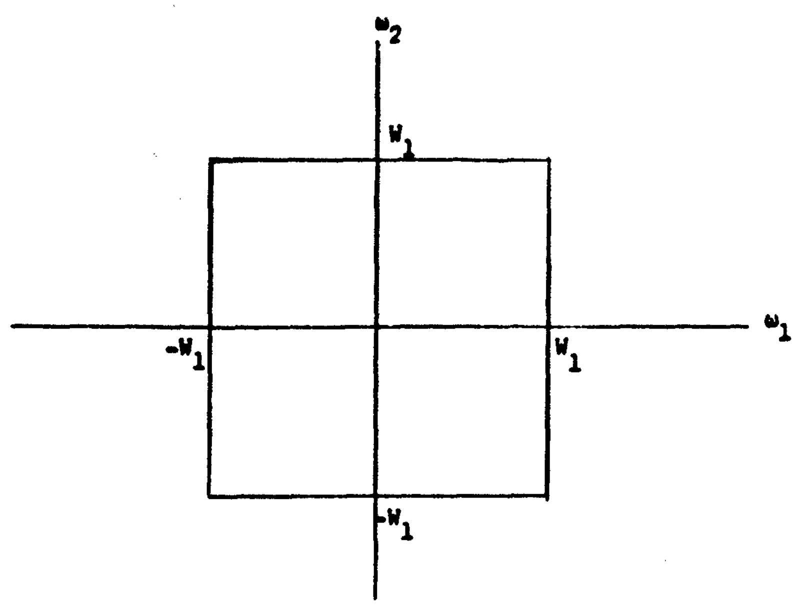
In this connection, consider the situation shown in Fig. 24. The nonlinear system $\underline{H}$ is cascaded after a linear ideal bandlimiting filter L_1 and before a linear ideal bandlimiting filter L_2. The filters L_1 and L_2 do not necessarily have the same passbands. We shall denote the transfer functions of these filters $L_1(\omega)$ and $L_2(\omega)$. We assume that the nonlinear system $\underline{H}$ can be characterized by an n^{th}-degree kernel $h_n(\tau_1,\ldots,\tau_n)$, and examine the effect of the ideal filters. If we think of the cascade combination of the filters L_1 and L_2 with $\underline{H}$ as a new nonlinear system, this new or over-all system will still be characterized by an n^{th}-degree kernel, which we denote $g_n(\tau_1,\ldots,\tau_n)$. The multidimensional Fourier transform of this kernel is

$$G_n(\omega_1,\ldots,\omega_n) = L_1(\omega_1)\ldots L_1(\omega_n)\, H_n(\omega_1,\ldots,\omega_n)\, L_2(\omega_1+\ldots+\omega_n), \qquad (75)$$

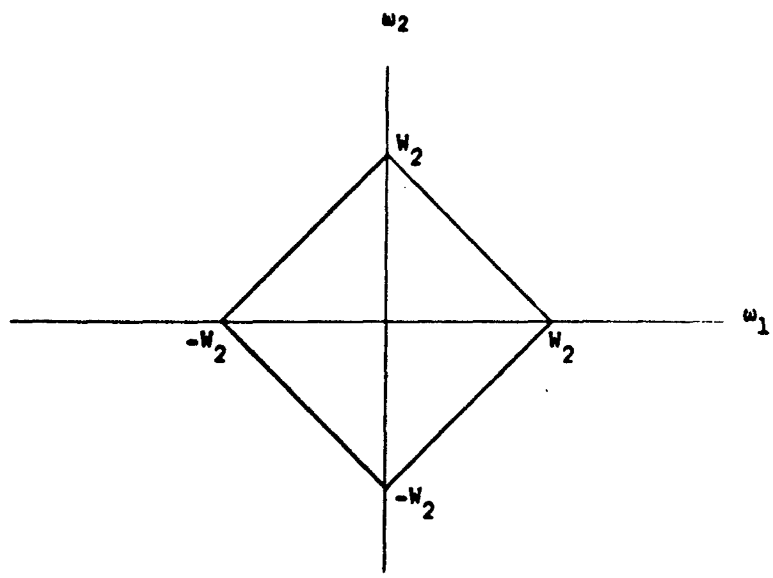
where $H_n(\omega_1,\ldots,\omega_n)$ is the transform of the kernel $h_n(\tau_1,\ldots,\tau_n)$.

For convenience, assume $n = 2$, and we may use the sketches in Fig. 25, in which we assume that L_1 and L_2 are lowpass, with bandwidths W_1 and W_2, respectively. The presence of L_1 then forces $G_2(\omega_1,\omega_2)$ to be nonzero only within the region shown in Fig. 25a, while the presence of L_2 limits $G_2(\omega_1,\omega_2)$ to the region shown in Fig. 25b.

We then define a system as bandlimited from the input if precascading an ideal bandlimiting filter has no effect on the over-all performance, and bandlimited at the output if postcascading an ideal bandlimiting filter has no effect on the over-all performance.



(a) Bandlimiter at the input.



(b) Bandlimiter at the output.

Fig. 25. Effects of cascading lowpass filters with a second-degree system.

Since linear time-invariant systems commute, a linear system that is bandlimited at the input is also bandlimited at the output; there is no difference in the two types of bandlimiting in this case. For nonlinear systems, however, the order of cascading cannot, in general, be interchanged without a corresponding change in over-all performance or character of the system, and it is then necessary to make the above-mentioned distinction between bandlimiting at the input and at the output. Consider again the second-degree case, with lowpass limiting, for which Fig. 26 applies. In Fig. 26, solid lines indicate the regions appropriate to input bandlimiting to (-W, W) or to (-2W, 2W);

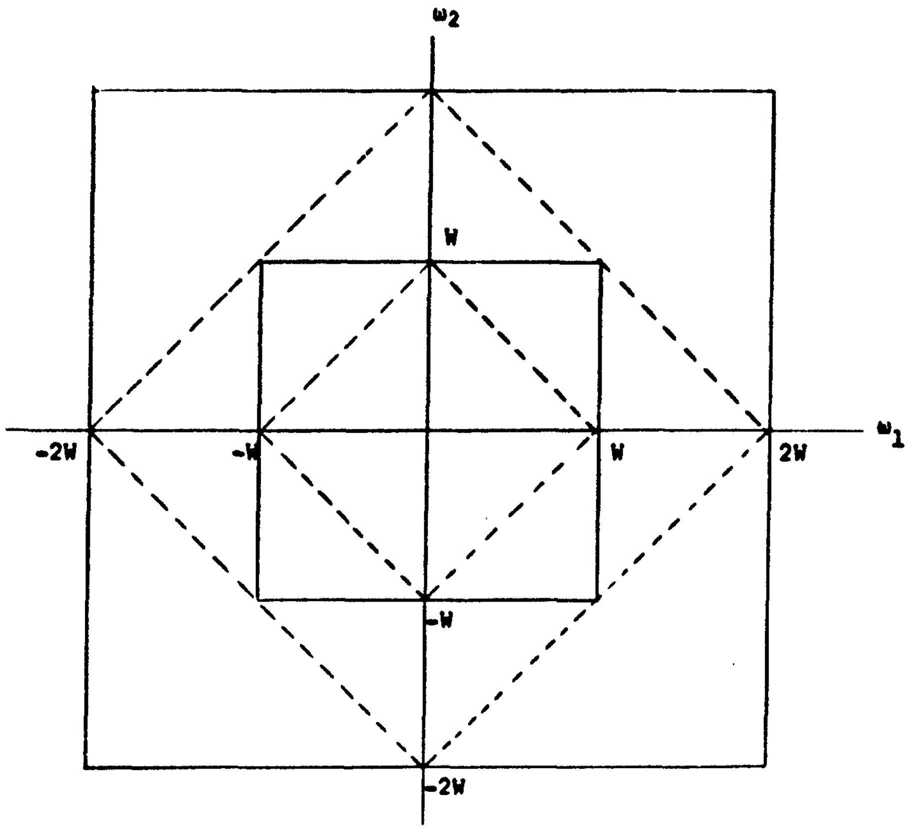


Fig. 26. Relation between input and output bandlimiting.

broken lines indicate the regions appropriate to output bandlimiting to (-W, W) or to (-2W, 2W). The following relationships are clear from the figure: (i) if postcascading an ideal filter with passband (-W, W) has no effect on over-all performance, then neither will precascading the same filter, that is, a lowpass filter at the output can be duplicated at the input without changing the over-all system; (ii) if precascading a filter with passband (-W, W) has no effect on over-all performance, then neither will postcascading a lowpass filter with passband (-2W, 2W).

These properties clearly generalize to the n^{th}-degree case and to bandpass, as well as lowpass, filtering. Thus, if a system is characterized by an n^{th}-degree kernel, we have the following situations: (i) if the system is bandlimited at the output, then it is bandlimited at the input also, and with at most the same bandwidth; (ii) if the system is bandlimited at the input, then it is bandlimited at the output also, but with at most n times the input bandwidth.

The highest frequency present in the output is at most n times the highest frequency present in the input, where n is the degree of the system. With reference to Fig. 24, if L_2 does not affect the output of the system, then $\underline{H}$ and L_2 commute.

a. Delay-Line Models for Bandlimited Systems

For nonlinear systems that are bandlimited at the input, we may take advantage of the special properties of systems with sampled inputs which were developed in

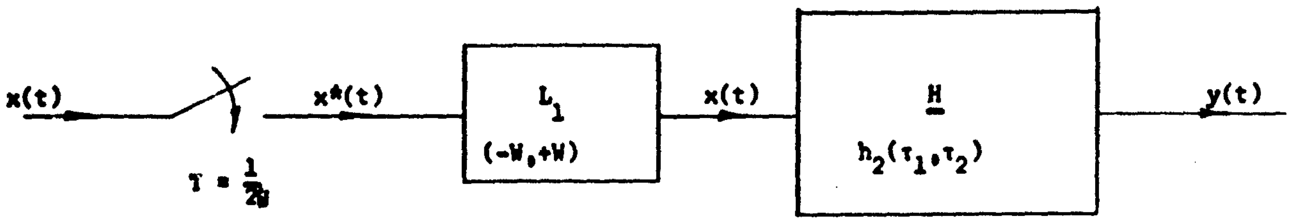


Fig. 27. Bandlimited system with bandlimited input.

Section III. Specifically, we may develop delay-line models for these systems, making use of the fact that the output for a sampled input depends only on the values of the kernels along certain lines in the domain of the kernels.

Consider the situation shown in Fig. 27. The nonlinear system is of second degree, that is, it is assumed to be characterized by the kernel $h_2(\tau_1, \tau_2)$ alone, and we assume that the system is bandlimited at the input to $(-W, W)$. Now we consider the precascading of an ideal sampler and an ideal lowpass filter with passband $(-W, W)$, as shown, so that the lowpass filter has no effect on the nonlinear system. For inputs $x(t)$ that are also bandlimited to $(-W, W)$, the cascade combination of the ideal sampler operating with the sampling interval $T = 1/2W$ and the lowpass filter with bandwidth W has no net effect. Thus, under these conditions, it is immaterial whether we present to the system the continuous input $x(t)$ or the sampled input $x^*(t)$.

In Section III we found that the output corresponding to the sampled input is given by

$$y(t) = \sum_{k_2=-\infty}^{+\infty} \sum_{k_1=-\infty}^{+\infty} x(k_1T)\, x(k_2T)\, h_2(t-k_1T, t-k_2T), \tag{76}$$

where $x(t)$ is the input, $y(t)$ the output, $h_2(\tau_1, \tau_2)$ the kernel of the system, and T the sampling interval of the input sampler.

Now let us substitute for the continuous system with the kernel $h_2(\tau_1, \tau_2)$, which we assume to be symmetrical, a system with the impulsive kernel

$$h_2'(\tau_1,\tau_2) = h_2(\tau_1,\tau_1)\, u_o(\tau_1-\tau_2) + 2 \sum_{k=1}^{\infty} h_2(\tau_1-kT,\tau_1)\, u_o(\tau_1-kT-\tau_2), \tag{77}$$

and present the continuous input $x(t)$ to this new system. In terms of Fig. 27, this is equivalent to ignoring the filter L_1 and associating the sampler with the system rather than with the input. The output $y(t)$ is still given by (76).

The kernel of (77) can now be realized by means of the delay-line model of Fig. 28, in which the linear systems have impulse responses given by

$$\begin{aligned} k_1(t) &= h_2(t,t) \\ k_2(t) &= 2h_2(t-T,t) \\ &\vdots \\ k_N(t) &= 2h_2(t-(N-1)T,t), \end{aligned} \tag{78}$$

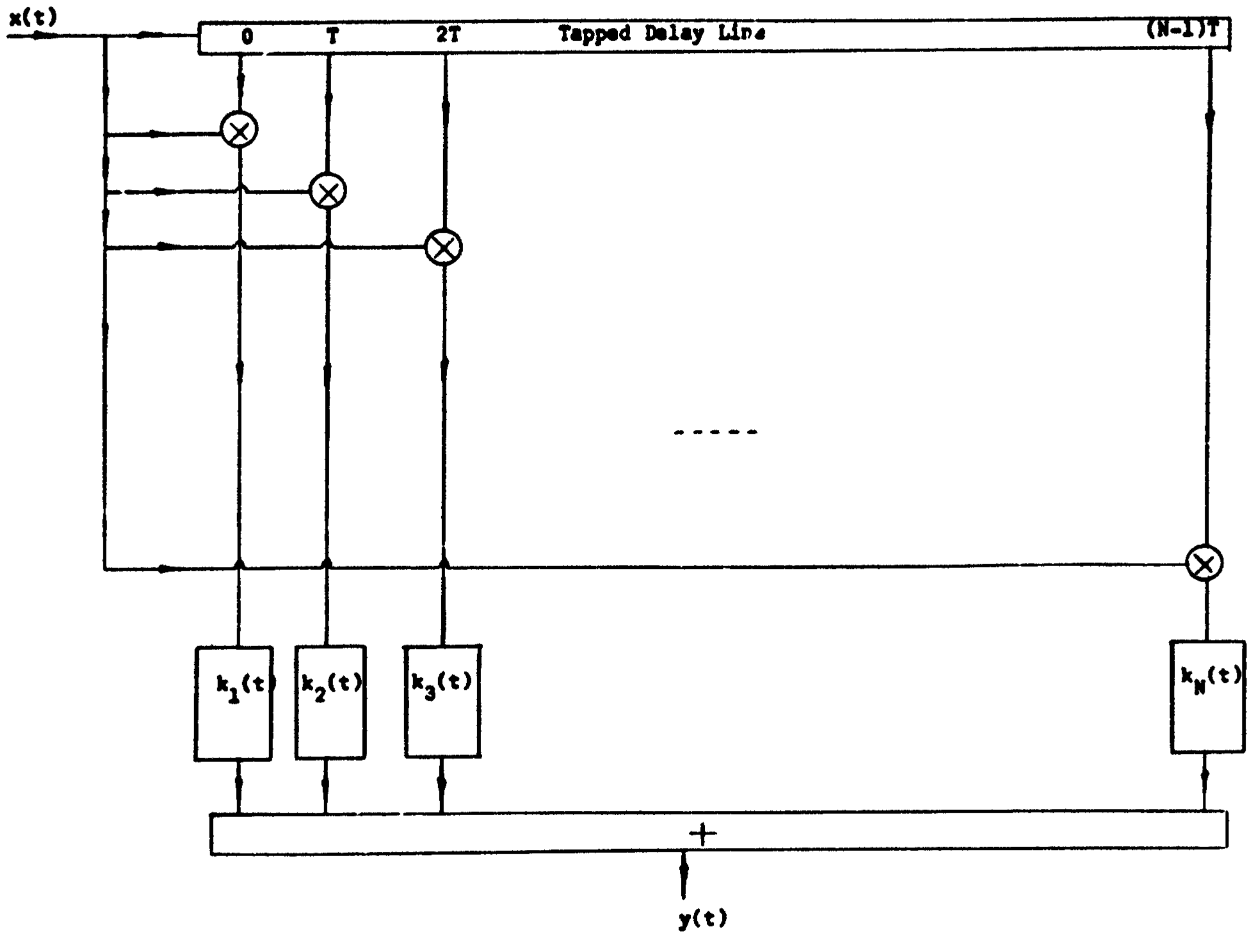


Fig. 28. Delay-line model for second-degree bandlimited system.

and we assume that $h_2(\tau_1, \tau_2)$ is zero outside the region shown in Fig. 29.

Note that we have assumed that the kernel is exactly bandlimited and essentially time-limited to the region of Fig. 29. It is impossible for the kernel to be both exactly time-limited and exactly bandlimited. We may also apply the results above to kernels that are essentially bandlimited to the prescribed band and which are time-limited. Practically, we shall be forced to deal with kernels that are both essentially bandlimited to the prescribed band and essentially time-limited to the prescribed region of the τ_1, τ_2 plane, in that almost all of the energy of the kernel lies within these regions.

Hence for a second-degree system that is essentially bandlimited at the input and time-limited to the region shown in Fig. 29, we have for essentially bandlimited inputs the realization of Fig. 28.

Delay-line models for higher-degree bandlimited systems can be developed similarly to this model for second-degree systems. The complexity increases very rapidly with

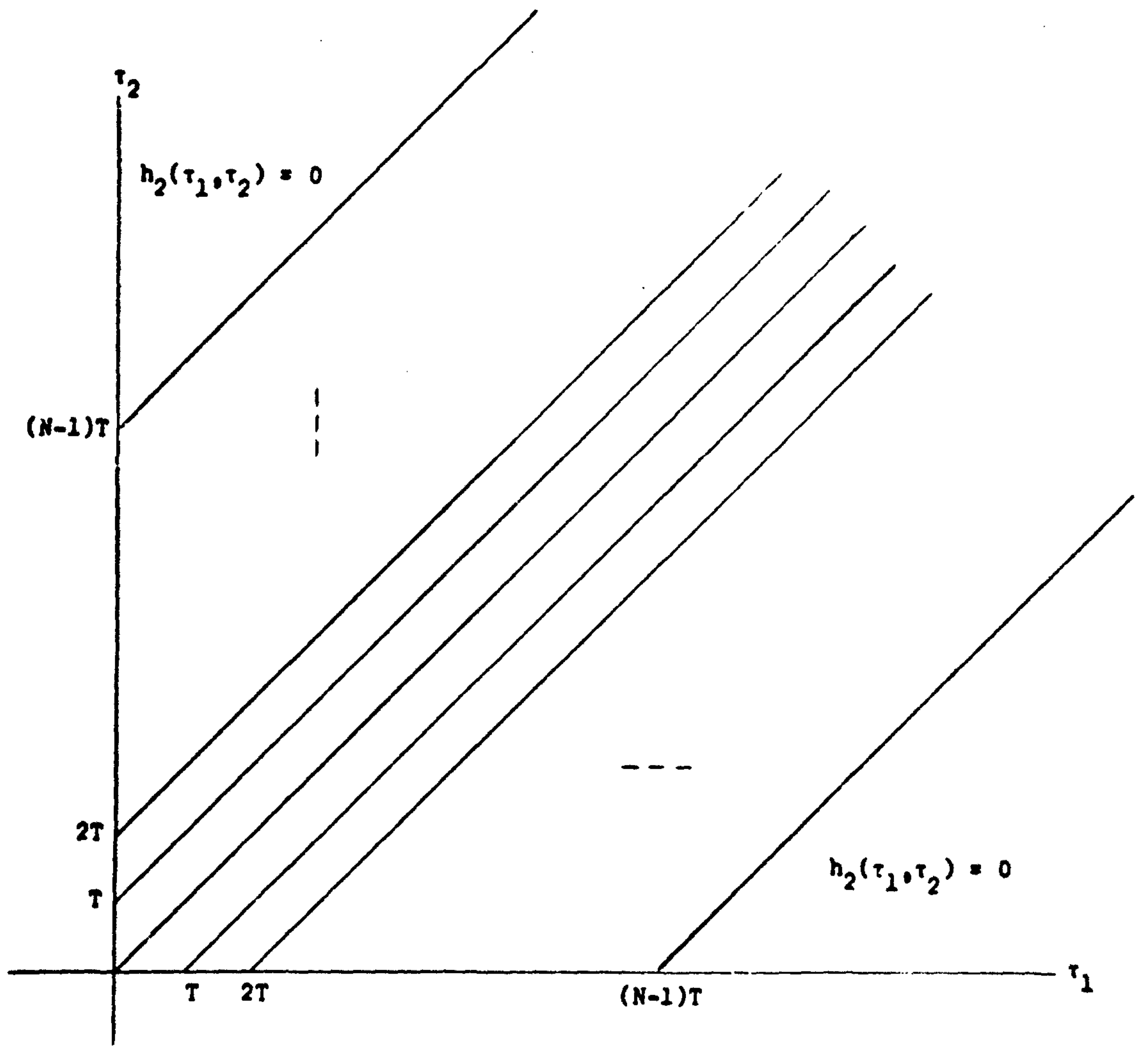


Fig. 29. Domain of the kernel of the delay-line model for a bandlimited second-degree system.

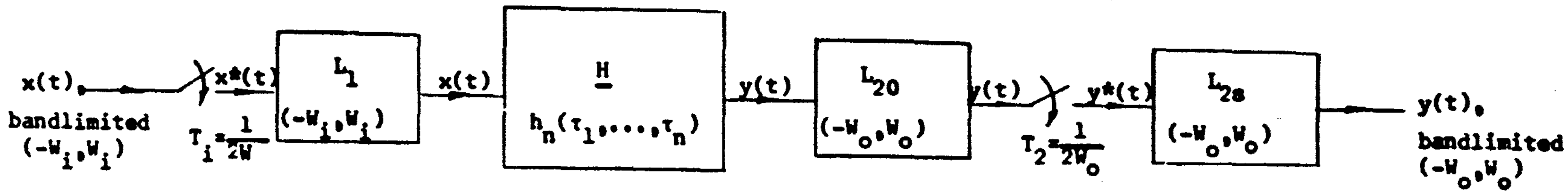


Fig. 30. Digital simulation of a bandlimited nonlinear system with a bandlimited input.

the degree of the system, however. For a given number of taps on the delay line, say N, the second-degree model requires N linear systems. The third-degree system, as can be seen from Eq. 59, would require that we form the product of the input with the square of the signal at each of the taps of the delay line, the product of the square of the input with each of the tap signals, and the product of the input with the input delayed by k units with the input delayed by j units for $k = 1, \ldots, N-1$ and $j = k, \ldots, N$; thus the number of linear systems needed in this case is $2(N-1) + \Sigma_{i=1}^{N-1}(N-i)$ or, on simplification, $1/2(N+2)(N-1)$. Thus the complexity grows roughly exponentially with the degree of the system for a given number of taps on the delay line.

b. Digital Simulation of Bandlimited Systems

In order to simulate a system on a digital computer, we must sample both the input and the output time functions. If the input is bandlimited and if the system is bandlimited at the input and output, then the situation shown in Fig. 30 is appropriate. We assume that $\underline{H}$ is bandlimited at the imput to $(-W_i, W_i)$ and at the output to $(-W_o, W_o)$, and the input $x(t)$ is also bandlimited to $(-W_i, W_i)$. The output $y(t)$ is then bandlimited to $(-W_o, W_o)$. We note from the discussion above that the output has a bandwidth of at most $W_o = nW_i$, where n is assumed to be the degree of the system.

Now, since we assume that $\underline{H}$ is bandlimited at the input and output, we may omit L_1 and L_{2o} in Fig. 30 without changing anything. We then simulate $\underline{H}$ by operating on $x^*(t)$ to obtain $y^*(t)$, and passing $y^*(t)$ through L_{2s} to reconstruct the continuous output $y(t)$. The computation that must be performed is

$$y(pT_o) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} x(k_1T_i) \ldots x(k_nT_i)\, h_n(pT_o-k_1T_i, \ldots, pT_o-k_nT_i) \qquad (79)$$

for discrete simulation of the bandlimited n^{th}-degree system. To the extent that the bandlimiting is ideal, this model performs the same operation as the continuous system. In the computation there will be an unavoidable error, because of quantization and roundoff.

V. TRANSFORM ANALYSIS OF NONLINEAR SAMPLED-DATA SYSTEMS

In Sections III and IV we discussed sampling in nonlinear systems and the simulation of continuous systems by sampled-data systems. The input-output relations for sampled-data systems have been developed; they are multidimensional convolution sums, which can be tedious to evaluate and give little insight into system properties. We shall now discuss a multidimensional z-transform and modified z-transform that will facilitate the study of nonlinear sampled-data systems.

The use of the multidimensional z-transform in the analysis of discrete nonlinear systems has been examined briefly by Alper.[23]

5.1 MULTIDIMENSIONAL Z-TRANSFORMS

Definition of the z-transform and Inverse z-transform. The multidimensional z-transform of a function $f(\tau_1,\ldots,\tau_n)$ may be defined as

$$F(z_1,\ldots,z_n) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T,\ldots,k_nT)\, z_1^{-k_1} \cdots z_n^{-k_n}. \tag{80}$$

We note that the z-transform of a function depends only on the values of the function at uniformly spaced sample points and not on the entire function.

We assume that $f(\tau_1,\ldots,\tau_n)$ has no impulsive components. If the summation of (80) converges at all, it will converge within some region defined by $\alpha_i \leq |z_i| \leq \beta_i, i = 1,\ldots,n$. A sufficient but not a necessary condition on $f(\tau_1,\ldots,\tau_n)$ such that its z-transform exist is that $f(\tau_1,\ldots,\tau_n)$ be absolutely integrable; in this case, the region of convergence will contain the multidimensional surface defined by the Cartesian product $\Gamma_1 x \Gamma_2 x \ldots x \Gamma_n$, where Γ_i is the unit circle in the z_i-plane. If a function $f(\tau_1,\ldots,\tau_n)$ has a Laplace transform, then it will also have a z-transform.

The value of the function at the sample points, $f(k_1T,\ldots,k_nT)$, may be obtained from the transform in a number of ways. A closed-form expression for the sample values in terms of the transform is given by the multidimensional inversion integral:

$$f(k_1T,\ldots,k_nT) = \left(\frac{1}{2\pi j}\right)^n \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} z_1^{k_1-1} \cdots z_n^{k_n-1} F(z_1,\ldots,z_n)\, dz_1 \ldots dz_n, \tag{81}$$

where Γ_i is an appropriate contour in the z_i-plane. For most of the functions considered here, Γ_i may be taken as the unit circle in the z_i-plane. The sample values may also be obtained as the coefficients in the series expansion of $F(z_1,\ldots,z_n)$ in $z_1^{-1},\ldots,z_n^{-1}$ about the origin in the $z_1,\ldots,z_n$-space. If $F(z_1,\ldots,z_n)$ is expressed as a ratio of polynomials in $z_1^{-1},\ldots,z_n^{-1}$, these coefficients may be found by division of the numerator polynomial by the denominator. Examples of the computation of the direct and inverse transforms are given below. A partial difference equation can also be found which gives a recursive method of computing the sample values from the transform; this method will be discussed in Section 6.2.

Properties of the Z-transform. The multidimensional z-transform has some properties which make it useful in the analysis of nonlinear sampled-data systems. We list some of these properties here; proofs are given in Appendix A.

If $f(\tau_1,\ldots,\tau_n) \longleftrightarrow F(z_1,\ldots,z_n)$, then

(5.a.1) $$f(\tau_1-b_1T,\ldots,\tau_n-b_nT) \longleftrightarrow z_1^{-b_1}\cdots z_n^{-b_n}F(z_1,\ldots,z_n),$$

where $b_1,\ldots,b_n$ are integers.

(5.a.2) $$e^{-a_1\tau_1}\cdots e^{-a_n\tau_n}f(\tau_1,\ldots,\tau_n) \longleftrightarrow F\left(e^{a_1T}z_1,\ldots,e^{a_nT}z_n\right)$$

(5.a.3) $$\tau_i f(\tau_1,\ldots,\tau_n) \longleftrightarrow -Tz_i\frac{\partial}{\partial z_i}F(z_1,\ldots,z_n), \qquad i = 1,\ldots n.$$

If in addition $g(\tau_1,\ldots,\tau_n) \longleftrightarrow G(z_1,\ldots,z_n)$, then the transform

(5.a.4) $$H(z_1,\ldots,z_n) = F(z_1,\ldots,z_n)\,G(z_1,\ldots,z_n)$$

corresponds to a function $h(\tau_1,\ldots,\tau_n)$ with sample values given by

$$h(p_1T,\ldots,p_nT) = \sum_{k_n=-\infty}^{\infty}\cdots\sum_{k_1=-\infty}^{\infty} f(k_1T,\ldots,k_nT)\,g(p_1T-k_1T,\ldots,p_nT-k_nT)$$

which is a multidimensional convolution expression.

Example 4

Let

$$f(\tau_1,\tau_2) = \begin{cases} 1 & \text{if } \begin{array}{l} 0 \le \tau_1 < 4 \\ 0 \le \tau_2 < 4 \end{array} \\ \\ 0 & \text{otherwise} \end{cases} \tag{82}$$

Then, taking $T = 1$, we have

$$F(z_1, z_2) = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} f(k_1, k_2)\, z_1^{-k_1} z_2^{-k_2}$$

$$= \sum_{k_2=0}^{3} \sum_{k_1=0}^{3} z_1^{-k_1} z_2^{-k_2} = \sum_{k_2=0}^{3} z_2^{-k_2} \sum_{k_1=0}^{3} z_1^{-k_1}$$

$$= \frac{1 - z_1^{-4}}{1 - z_1^{-1}} \cdot \frac{1 - z_2^{-4}}{1 - z_2^{-1}} = \frac{1 - z_1^{-4} - z_2^{-4} + z_1^{-4} z_2^{-4}}{1 - z_1^{-1} - z_2^{-1} + z_1^{-1} z_2^{-1}}. \tag{83}$$

Dividing the numerator by the denominator, we find the finite series with which we began:

$$F(z_1, z_2) = 1 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1} + z_1^{-2} + z_2^{-2} + z_1^{-1} z_2^{-2} + z_1^{-2} z_2^{-1}$$
$$+ z_1^{-3} + z_2^{-3} + z_1^{-1} z_2^{-3} + z_1^{-3} z_2^{-1} + z_1^{-2} z_2^{-2} + z_1^{-2} z_2^{-3}$$
$$+ z_1^{-3} z_2^{-2} + z_1^{-3} z_2^{-3}. \tag{84}$$

The coefficients of this expansion give the sample values of $f(\tau_1, \tau_2)$.

<u>Example 5</u>

Let

$$f(\tau_1, \tau_2) = \begin{cases} e^{-\tau_1} e^{-3\tau_2}\left(1 - e^{-2\tau^*}\right), & \tau_1 \geq 0,\ \tau_2 \geq 0 \\ 0 & \tau_1 < 0 \text{ or } \tau_2 < 0 \end{cases} \tag{85}$$

where $\tau^* = \min(\tau_1, \tau_2)$. Consider first the function

$$f_1(\tau_1, \tau_2) = \begin{cases} \left(1 - e^{-2\tau^*}\right) & \tau_1 \geq 0,\ \tau_2 \geq 0 \\ 0 & \tau_1 < 0 \text{ or } \tau_2 < 0 \end{cases} \tag{86}$$

Then we have for the z-transform, with sampling interval T,

$$F_1(z_1, z_2) = \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \left(1 - e^{-2k^*T}\right) z_1^{-k_1} z_2^{-k_2}, \qquad k^* = \min(k_1, k_2)$$

$$= \sum_{k=0}^{\infty} (1 - e^{-2kT})(z_1 z_2)^{-k} + \sum\sum_{k_1 > k_2 \geq 0} \left(1 - e^{-2k_2T}\right) z_1^{-k_1} z_2^{-k_2}$$

$$+ \sum\sum_{0 \leq k_1 < k_2} \left(1 - e^{-2k_1T}\right) z_1^{-k_1} z_2^{-k_2}. \tag{87}$$

In the second summation, make the change of index $k_1 = k_2 + k_1'$; then this sum becomes

$$\sum_{k_1'=1}^{\infty} \sum_{k_2=0}^{\infty} \left(1 - e^{-2k_2T}\right) z_1^{-(k_2+k_1')} z_2^{-k_2} = \sum_{k_1'=1}^{\infty} z_1^{-k_1'} \cdot \sum_{k_2=0}^{\infty} \left(1 - e^{-2k_2T}\right)(z_1 z_2)^{-k_2}. \tag{88}$$

The last sum, on $0 \leq k_1 < k_2$, can be treated similarly. We then have

$$F_1(z_1, z_2) = \left[\sum_{k=0}^{\infty} (1 - e^{-2kT})(z_1 z_2)^{-k}\right]\left[1 + \sum_{k=1}^{\infty} z_1^{-k} + \sum_{k=1}^{\infty} z_2^{-k}\right] \tag{89}$$

or, evaluating these sums,

$$F_1(z_1, z_2) = \left[\frac{1}{1 - z_1^{-1} z_2^{-1}} - \frac{1}{1 - e^{-2T} z_1^{-1} z_2^{-1}}\right]\left[\frac{1}{1 - z_1^{-1}} + \frac{1}{1 - z_2^{-1}} - 1\right]$$

$$= \frac{(1 - e^{-2T})\, z_1^{-1} z_2^{-1}}{\left(1 - e^{-2T} z_1^{-1} z_2^{-1}\right)\left(1 - z_1^{-1}\right)\left(1 - z_2^{-1}\right)} \tag{90}$$

after some simplification. Now using property (5. a. 2) for the transform of $f(\tau_1, \tau_2)$ we have

$$F(z_1, z_2) = \frac{(e^{-4T} - e^{-6T})\, z_1^{-1} z_2^{-1}}{\left(1 - e^{-6T} z_1^{-1} z_2^{-1}\right)\left(1 - e^{-T} z_1^{-1}\right)\left(1 - e^{-3T} z_2^{-1}\right)} . \tag{91}$$

We may make use of the inversion integral for the z-transform to obta n from $F(z_1, z_2)$

the value of any of the samples of $f(\tau_1, \tau_2)$. Choosing $f(T,T)$, we have

$$f(T,T) = \left(\frac{1}{2\pi j}\right)^2 \oint_\Gamma \oint_\Gamma \frac{(e^{-4T} - e^{-6T})\, z_1 z_2}{\left(z_1 z_2 - e^{-6T}\right)\left(z_1 - e^{-T}\right)\left(z_2 - e^{-3T}\right)} dz_1 dz_2$$

$$= (e^{-4T} - e^{-6T}) \frac{1}{2\pi j} \oint \frac{z_2}{\left(z_2 - e^{-5T}\right)\left(z_2 - e^{-3T}\right)} - \frac{e^{-5T}}{\left(z_2 - e^{-5T}\right)\left(z_2 - e^{-3T}\right)} dz_2$$

$$= (e^{-4T} - e^{-6T}) \frac{1}{2\pi j} \oint \frac{dz_2}{\left(z_2 - e^{-3T}\right)} = (e^{-4T} - e^{-6T}), \tag{92}$$

where we have evaluated first the integral on z_1, then the integral on z_2, using the Cauchy residue theorem, with Γ the unit circle in each of the integrations.

5.2 MODIFIED Z-TRANSFORMS

The z-transform of a function depends only on the values of the function at uniformly spaced sample points. Often we would like to focus attention on the sample points, but are nevertheless interested in the remainder of the function; in such cases the modified z-transform is useful. The multidimensional modified z-transform of a function $f(\tau_1, \ldots, \tau_n)$ may be defined as

$$F_m(z_1, m_1; \ldots ; z_n, m_n) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f[k_1 T - (1-m_1)T, \ldots, k_n T - (1-m_n)T]\, z_1^{-k_1} \cdots z_n^{-k_n}, \tag{93}$$

where $0 \leq m_i < 1$, $i = 1, \ldots, n$. Choosing a value for each of the m_i amounts to shifting the function along each of the axes in its domain of definition before taking the z-transform. Existence of the transform, that is, convergence of (82), is insured under the same conditions as given above for the ordinary z-transform.

The modified z-transform may be inverted to regain the function $f(\tau_1, \ldots, \tau_n)$ by means of the multidimensional inversion integral

$$f[(k_1 - 1 + m_1)T, \ldots, (k_n - 1 + m_n)T] =$$

$$\left(\frac{1}{2\pi j}\right)^n \oint_{\Gamma_n} \cdots \oint_{\Gamma_n} z_1^{k_1 - 1} \cdots z_n^{k_n - 1} F_m(z_1, m_1; \ldots ; z_n, m_n)\, dz_1 \ldots dz_n, \tag{94}$$

where Γ_i is an appropriate contour in the z_i-plane. As for the inverse z-transform (81),

Γ_i can be taken as the unit circle in the z_i-plane for most of the functions considered here. Note that $m_1, \ldots, m_n$ are treated as parameters in the inversion integral. If the modified z-transform $F_m(z_1, m_1; \ldots; z_n, m_n)$ is expanded in a series in $z_1^{-1}, \ldots, z_n^{-1}$ about the origin, the coefficients of the expansion will be functions of $m_1, \ldots, m_n$, $0 \leq m_i < 1$, $i = 1, \ldots, n$, and will give the function $f(\tau_1, \ldots, \tau_n)$ in the corresponding multidimensional sampling interval in the $(\tau_1, \ldots, \tau_n)$-space. Examples of the computation of the direct and inverse modified z-transform will be given below.

<u>Properties of the Modified z-transform.</u> Some useful properties of the modified z-transform are listed below; proofs are given in Appendix B.

If $f(\tau_1, \ldots, \tau_n) \longleftrightarrow F_m(z_1, m_1; \ldots; z_n, m_n)$, then

(5. b. 1) $$f(\tau_1 - b_1 T, \ldots, \tau_n - b_n T) \longleftrightarrow z_1^{-b_1} \ldots z_n^{-b_n} F_m(z_1, m_1; \ldots; z_n, m_n),$$

where $b_1, \ldots, b_n$ are integers. For shifts not equal to an integral number of sampling intervals, we have

$$f(\tau_1 - \Delta_1 T, \ldots, \tau_n - \Delta_n T) \longleftrightarrow \begin{cases} z_1^{-1} \ldots z_n^{-1} F_m(z_1, m_1 + 1 - \Delta_1; \ldots; z_n, m_n + 1 - \Delta_n) \\ \qquad\qquad 0 \leq m_i < \Delta_i \quad i = 1, \ldots, n \\ F_m(z_1, m_1 - \Delta_1; \ldots; z_n, m_n - \Delta_n) \\ \qquad\qquad \Delta_i \leq m_i < 1 \quad i = 1, \ldots, n \end{cases}$$

(5. b. 2) $$e^{-a_1 T} \ldots e^{-a_n T} f(\tau_1, \ldots, \tau_n) \longleftrightarrow$$
$$e^{-a_1 T(m_1 - 1)} \ldots e^{-a_n T(m_n - 1)} F\left(e^{a_1 T} z_1, m_1; \ldots; e^{a_n T} z_n, m_n\right)$$

(5. b. 3) $$\tau_i f(\tau_1, \ldots, \tau_n) \longleftrightarrow T\Big[(m_i - 1) F_m(z_1, m_1; \ldots; z_n, m_n)$$
$$- z_i \frac{\partial}{\partial z_i} F_m(z_1, m_1; \ldots; z_n, m_n)\Big] \qquad i = 1, \ldots n$$

If in addition $f(\tau_1, \ldots, \tau_n) \longleftrightarrow F(z_1, \ldots, z_n)$, then

(5. b. 4) $$F(z_1, \ldots, z_n) = z_1 \ldots z_n F_m(z_1, m; \ldots; z_n, m_n)\Big|_{m_1 = \ldots = m_n = 0}$$

For continuous $f(\tau_1, \ldots, \tau_n)$, we have also

$$F(z_1, \ldots, z_n) = F_m(z_1 m_1; \ldots; z_n, m_n)\Big|_{m_1 = \ldots = m_n = 1}$$

Example 6

Let

$$f(\tau_1, \tau_2) = \begin{cases} e^{-\tau_1} e^{-3\tau_2}\left(1 - e^{-2\tau^*}\right) & \tau_1 \geq 0,\ \tau_2 \geq 0 \\ 0 & \tau_1 < 0 \text{ or } \tau_2 < 0 \end{cases} \tag{95}$$

where $\tau^* = \min(\tau_1, \tau_2)$, as in Example 5. Again consider first the function

$$f_1(\tau_1, \tau_2) = \begin{cases} \left(1 - e^{-2\tau^*}\right) & \tau_1 \geq 0,\ \tau_2 \geq 0 \\ 0 & \tau_1 < 0 \text{ or } \tau_2 < 0 \end{cases} \tag{96}$$

Then we have

$$F_{1m}(z_1, m_1; z_2, m_2) = \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \left(1 - e^{-2T(k^* + m^* - 1)}\right) z_1^{-k_1} z_2^{-k_2}, \tag{97}$$

where $k^* = \min(k_1, k_2)$, and $m^* = \min(m_1, m_2)$. By making use of the same techniques as in Example 4 (see Appendix B), this sum can be written

$$F_{1m}(z_1, m_1; z_2, m_2) = \frac{1 - e^{-2T(m^*-1)} - z_1^{-1} z_2^{-1}\left(e^{-2T} - e^{-2T(m^*-1)}\right)}{\left(1 - e^{-2T} z_1^{-1} z_2^{-1}\right)\left(1 - z_1^{-1}\right)\left(1 - z_2^{-1}\right)}. \tag{98}$$

Then use of property (5. b. 2) for modified z-transforms gives the desired transform:

$$F_m(z_1, m_1; z_2, m_2) = e^{-T(m_1-1)}\, e^{-3T(m_2-1)}$$

$$\cdot \left[\frac{1 - e^{-2T(m^*-1)} - z_1^{-1} z_2^{-1}\left(e^{-6T} - e^{-2Tm^*}\, e^{-2T}\right)}{\left(1 - e^{-6T} z_1^{-1} z_2^{-1}\right)\left(1 - e^{-T} z_1^{-1}\right)\left(1 - e^{-3T} z_2^{-1}\right)}\right]. \tag{99}$$

In order to obtain the inverse, consider first the denominator of the term in brackets; we may write this as a polynomial in z_1^{-1} and z_2^{-1}:

$$1 - e^{-T} z_1^{-1} - e^{-3T} z_2^{-1} + (e^{-4T} - e^{-6T}) z_1^{-1} z_2^{-1} + e^{-9T} z_1^{-1} z_2^{-2} + e^{-7T} z_1^{-2} z_2^{-1}$$
$$- e^{-10T} z_1^{-2} z_2^{-2}. \tag{100}$$

Dividing this into unity provides an expansion, which after juggling with the operations indicated in the numerator of the bracketed term in the transform and combining, gives the series expansion of the modified z-transform $F_m(z_1, m_1; \ldots; z_n, m_n)$. The first few terms of this expansion are

$$F_m(z_1, m_1; \ldots; z_n, m_n) = e^{-T(m_1-1)} e^{-3T(m_2-1)} [(1-e^{-2T(m^*-1)}) + (e^{-T} - e^{-2Tm^*+T}) z_1^{-1}$$
$$+ (e^{-3T} - e^{-2Tm^*-T}) z_2^{-1} + (e^{-2T} - e^{-2Tm^*}) z_1^{-2}$$
$$+ (e^{-6T} - e^{-2Tm^*-4T}) z_2^{-2} + (e^{-4T} - e^{-2Tm^*-4T}) z_1^{-1} z_2^{-1}$$
$$+ \ldots] \tag{101}$$

The coefficients of each term in this expansion display the function $f(\tau_1, \tau_2)$ as a function of m_1, m_2 in the two-dimensional sampling interval corresponding to that term.

5.3 SOME PROPERTIES OF THE TRANSFORMS OF CAUSAL FUNCTIONS

The multidimensional z-transform and the modified z-transform of functions $f(\tau_1, \ldots, \tau_n)$ which are "causal," that is, which are zero when any of the arguments $\tau_1, \ldots, \tau_n$ are negative, have some special properties. We list some of these below; proofs are given in Appendix C.

Initial and Final Value. If $f(\tau_1, \ldots, \tau_n) \longleftrightarrow F(z_1, \ldots, z_n)$, and $f(\tau_1, \ldots, \tau_n) = 0$ for any of the τ_i less than zero, then

(5. c. 1) $\quad \lim\limits_{\tau_i \to 0} f(\tau_1, \ldots, \tau_n) \longleftrightarrow \lim\limits_{z_i \to \infty} F(z_1, \ldots, z_n), \qquad i = 1, \ldots n.$

(5. c. 2) $\quad \lim\limits_{\tau_i \to \infty} f(\tau_1, \ldots, \tau_n) \longleftrightarrow \lim\limits_{z_i \to 1} (z_i - 1) F(z_1, \ldots, z_n), \qquad i = 1, \ldots n.$

Relation to Laplace Transforms. The z-transform $F(z_1,\ldots,z_n)$ or modified z-transform $F_m(z_1,m_1;\ldots;z_n,m_n)$ of a causal function $f(\tau_1,\ldots,\tau_n)$ can be found directly from the Laplace transform $F_L(s_1,\ldots s_n)$ of the function by means of the following integrals. A justification of the integrals is given in Appendix C; an example of their use is given below.

$$F(z_1,\ldots,z_n)\Bigg|_{\substack{z_i=e^{s_iT}\\ i=1,\ldots,n}} = \left(\frac{1}{2\pi j}\right)^n \int_{\sigma_n-j\infty}^{\sigma_n+j\infty}\cdots\int_{\sigma_1-j\infty}^{\sigma_1+j\infty} \frac{F_L(\nu_1,\ldots,\nu_n)d\nu_1\ldots d\nu_n}{(1-e^{-T(s_1-\nu_1)})\ldots(1-e^{-T(s_n-\nu_n)})} \tag{102}$$

$$F_m(z_1,m_1;\ldots;z_n,m_n)\Bigg|_{\substack{z_i=e^{s_iT}\\ i=1,\ldots,n}} =$$

$$z_1^{-1}\ldots z_n^{-1}\left(\frac{1}{2\pi j}\right)^n \int_{\sigma_n-j\infty}^{\sigma_n+j\infty}\cdots\int_{\sigma_1-j\infty}^{\sigma_1+j\infty} \frac{F_L(\nu_1,\ldots,\nu_n)e^{m_1\nu_1T}\ldots e^{m_n\nu_nT}}{(1-e^{-T(s_1-\nu_1)})\ldots(1-e^{-T(s_n-\nu_n)})}\,d\nu_1\cdots d\nu_n \tag{103}$$

For most functions considered here we may take $\sigma_1 = \ldots = \sigma_n = 0$. Special care must be exercised in the use of these expressions for a function $f(\tau_1,\ldots,\tau_n)$ which is not continuous; at jump discontinuities of $f(\tau_1,\ldots,\tau_n)$ these expressions assume that $f(\tau_1,\ldots,\tau_n)$ is defined to be the average value at the discontinuity.

Example 7

Let $f(\tau_1,\tau_2)$ be the function whose Laplace transform is

$$F_L(s_1,s_2) = \frac{2}{(s_1+1)(s_2+3)(s_1+s_2+6)} \tag{104}$$

We may find the z-transform of $f(\tau_1,\tau_2)$, with a sampling interval T, directly from (104) as follows.

$$F(z_1,z_2)\Big|_{\substack{z_1=e^{s_1T}\\ z_2=e^{s_2T}}} = \left(\frac{1}{2\pi j}\right)^2 \int_{-j\infty}^{j\infty}\int_{-j\infty}^{j\infty} \frac{2dv_1dv_2}{(v_1+1)(v_2+3)(v_1+v_2+6)(1-e^{-(s_1-v_1)T})(1-e^{-(s_2-v_2)T})}$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{2dv_2}{(v_2+3)(1-e^{-(s_2-v_2)T})(v_2+5)}\left[\frac{1}{1-e^{-(s_1+1)T}} - \frac{1}{1-e^{-(s_1+v_2+6)T}}\right]$$

$$= \frac{1}{2\pi j}\,\frac{e^{-T}e^{-s_1T}}{1-e^{-T}e^{-s_1T}}\int_{-j\infty}^{j\infty} \frac{2(1-e^{-5T}e^{-v_2T})dv_2}{(v_2+3)(1-e^{-(s_2-v_2)T})(v_2+5)(1-e^{-(s_1+v_2+6)T})}$$

$$= \frac{e^{-T}e^{-s_1T}}{1-e^{-T}e^{-s_1T}}\;\frac{1}{1-e^{-6T}e^{-s_1T}e^{-s_2T}}\;\frac{1}{T}\sum_{k=-\infty}^{\infty} G\left(s_2+j\,\frac{2\pi k}{T}\right) \tag{105}$$

where

$$G(s_2) = \frac{(1-e^{-5T}e^{-s_2T})}{(s_2+3)(s_2+5)} \tag{106}$$

and we have evaluated first the integral with respect to ν_1, then the integral with respect to ν_2, using the Residue Theorem and closing the contour of integration in the left-plane for ν_1 and the right half-plane for ν_2. We recognize the summation as the Laplace transform of the sampled version of the function whose Laplace transform is G(s). That is, if $g(t) \longleftrightarrow G(s)$ and $g^*(t)$ is the sampled version of g(t)

$$g^*(t) = \sum_{k=-\infty}^{\infty} g(kT)u_o(t-kT), \tag{107}$$

then $G^*(s)$ is given by the summation in the expression above for $F(z_1, z_2)$. Evaluation of this sum at $e^{s_2T} = z_2$ will then yield the z-transform of g(t). This z-transform is readily obtained from a partial fraction expansion of G(s) or use of (104) for $n = 1$ as

$$\frac{z_2^{-1}(e^{-3T} - e^{-5T})}{\left(1 - e^{-3T} z_2^{-1}\right)} \tag{108}$$

and hence we have the desired z-transform

$$F(z_1, z_2) = \frac{(e^{-4T} - e^{-6T})\, z_1^{-1} z_2^{-1}}{\left(1 - e^{-T} z_1^{-1}\right)\left(1 - e^{-3T} z_2^{-1}\right)\left(1 - e^{-6T} z_1^{-1} z_2^{-1}\right)} . \tag{109}$$

5.4 APPLICATION TO NONLINEAR SYSTEMS

A nonlinear system with sampled input and output is shown in Fig 31. The sampling interval at the input is T_i and that at the output is T_o. The input is x(t) and the sampled

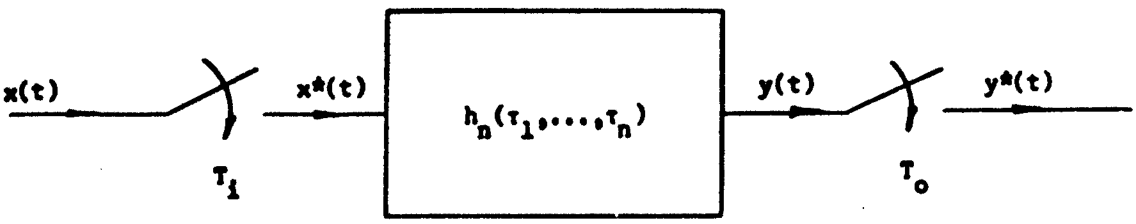


Fig. 31. Nonlinear system with sampled input and output.

input is $x^*(t)$; the output is y(t) and the sampled output is $y^*(t)$. The system is characterized by the kernel $h_n(\tau_1, \ldots, \tau_n)$, which we assume contains no impulsive components. This situation was discussed in Section III, where the input-output relation was given as

$$y(pT) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} x(k_1T) \ldots x(k_nT)\, h_n(pT-k_1T, \ldots, pT-k_nT) \tag{110}$$

and we assume, initially, that $T_i = T_o = T$.

Input-Output Computation. Although (110) is not quite a multidimensional convolution sum of the form given in property (5. a. 4), we may make use of an artifice suggested by George[25] for continuous systems, introducing the auxiliary function

$$y_{(n)}(p_1T, \ldots, p_nT) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} x(k_1T) \ldots x(k_nT)\, h_n(p_1T-k_1T, \ldots, p_nT-k_nT) \tag{111}$$

from which the sample values can be found:

$$y_{(n)}(p_1T, \ldots, p_nT)\Big|_{p_1=\ldots=p_n=p} = y(pT) \tag{112}$$

Using property (5. a. 4), we find that the transform of the auxiliary function is given by

$$Y_{(n)}(z_1, \ldots, z_n) = H_n(z_1, \ldots, z_n)X(z_1) \ldots X(z_n), \tag{113}$$

where $H_n(z_1, \ldots, z_n)$ is the multidimensional z-transform of the kernel, and $X(z)$ is the one-dimensional z-transform of the input.

If we do not sample the output, we have in place of (111)

$$y_{(n)}(t_1, \ldots, t_n) = \sum_{k_n=-\infty}^{\infty} \ldots \sum_{k_1=-\infty}^{\infty} x(k_1T) \ldots x(k_nT)\, h_n(t_1-k_1T, \ldots, t_n-k_nT). \tag{114}$$

Taking the modified z-transform then gives

$$Y_{(n)m}(z_1, m_1; \ldots; z_n, m_n) = H_{nm}(z_1, m_1; \ldots; z_n, m_n)\, X(z_1) \ldots X(z_n). \tag{115}$$

Association of Variables. As shown in Appendix D, the one-dimensional z-transform of the output, $Y(z)$, can be found from $Y_{(n)}(z_1, \ldots z_n)$ by use of the following integral.

$$Y(z) = \left(\frac{1}{2\pi j}\right)^{n-1} \oint_{\Gamma_{n-1}} \ldots \oint_{\Gamma_1} (w_1 w_2 \ldots w_{n-1})^{-1}\, Y_{(n)}\left(w_1, \frac{w_2}{w_1}, \ldots, \frac{z}{w_{n-1}}\right) dw_1 \ldots dw_{n-1}. \tag{116}$$

For continuous outputs, the corresponding expression in terms of the modified z-transform is

$$Y_m(z, m) = \left(\frac{1}{2\pi j}\right)^{n-1} \oint_{\Gamma_{n-1}} \ldots \oint_{\Gamma_1} (w_1 w_2 \ldots w_{n-1})^{-1}\, Y_{(n)m}\left(w_1, \frac{w_2}{w_1}, \ldots, \frac{z}{w_{n-1}}\right) \cdot dw_1 \ldots dw_{n-1}. \tag{117}$$

This procedure is analogous to the association of variables given by George for the continuous case, and, as in the continuous case, can be carried out as an inspection technique for simple functions. The basis of the inspection technique is the replacement

$$A(z_1 z_2)\, \frac{1}{\left(1 - e^{-aT} z_1^{-1}\right)}\, \frac{1}{\left(1 - e^{-bT} z_2^{-1}\right)} \longrightarrow A(z)\, \frac{1}{1 - e^{-(a+b)T}\, z^{-1}} \tag{118}$$

which is derived in the Appendix D. An example of the use of this technique is given below.

Different Sampling Rates at the Input and Output. If the output sampling interval is not equal to the input sampling interval, Eq. 110 becomes

$$y(pT_o) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} x(k_1T_i) \ldots x(k_nT_i)\, h_n(pT_o-k_1T_i, \ldots, pT_o-k_nT_i). \quad (119)$$

Let $T_i = rT_o$, where r is a positive integer. Then we have

$$y(pT_o) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} x(k_1rT_o) \ldots x(k_nrT_o)\, h_n(pT_o-k_1rT_o, \ldots, pT_o-k_nrT_o). \quad (120)$$

Forming the auxiliary function $y_{(n)}(p_1T_o, \ldots, p_nT_o)$ and then the z-transform $Y_{(n)}(z_1, \ldots, z_n)$, we have

$$Y_{(n)}(z_1, \ldots, z_n) = H_n(z_1, \ldots, z_n)\, X\left(z_1^r\right) \ldots X\left(z_n^r\right). \quad (121)$$

Interconnections of Systems. We consider here the interconnection of nonlinear sampled-data systems $\underline{H}$ and $\underline{K}$ when we assume that the systems are characterized by the Volterra kernels $h_n(\tau_1, \ldots, \tau_n)$, $n = 0, 1, \ldots, N_h$, and $k_n(\tau_1, \ldots, \tau_n)$, $n = 0, 1, \ldots, N_k$ before the input and the output are sampled. We will denote the interconnection of $\underline{H}$

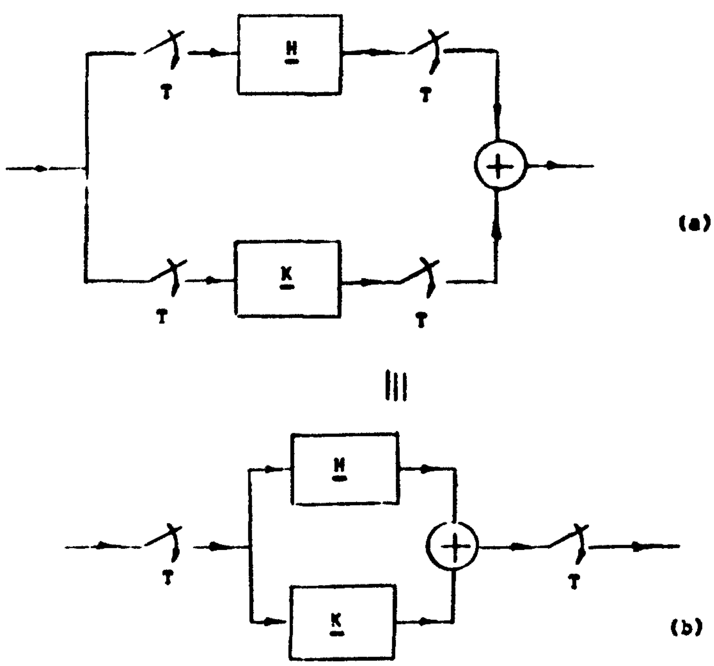


Fig. 32. Sum of sampled-data systems.

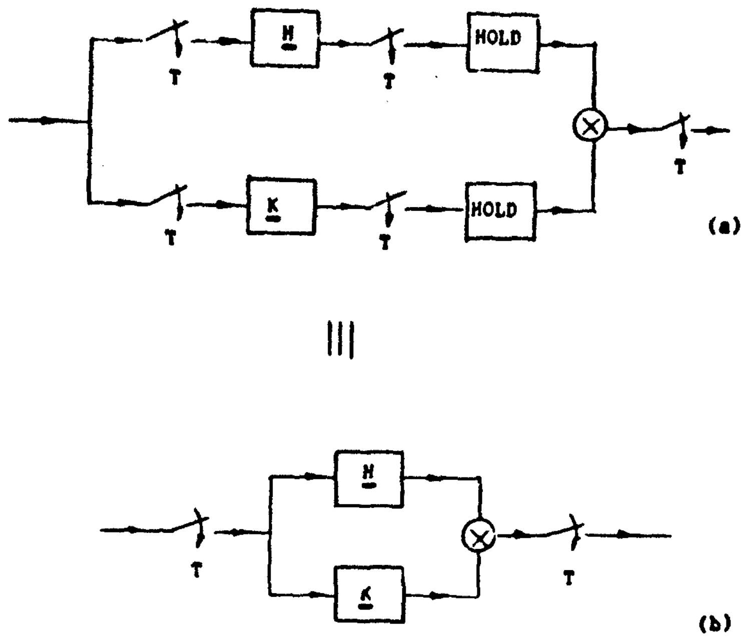


Fig. 33. Product of sampled-data systems.

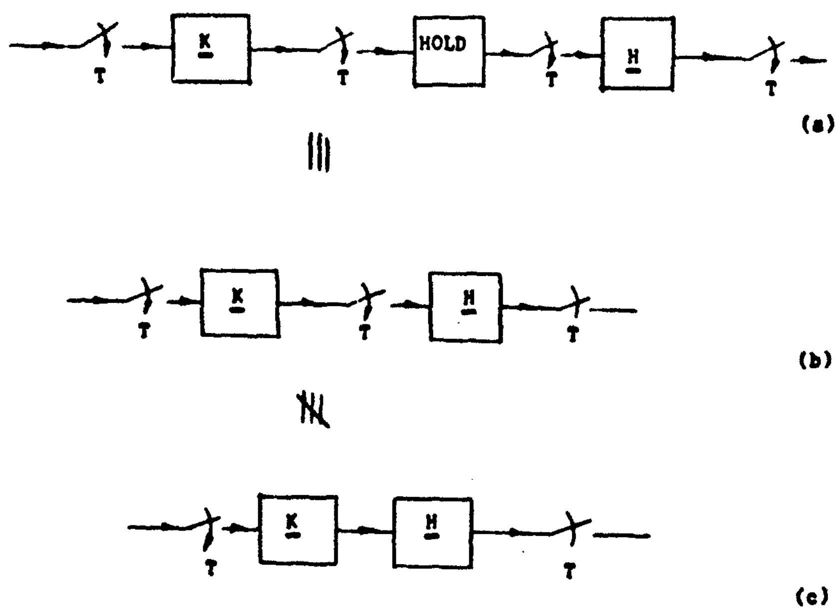


Fig. 34. Cascade combination of sampled-data systems.

and $\underline{K}$ by $\underline{G}$, with the Volterra kernels $g_n(\tau_1, \ldots, \tau_n)$, $n = 0, 1, \ldots, N_g$, before the input and output are sampled.

Consider first the addition of $\underline{H}$ and $\underline{K}$, as shown in Fig. 32. It is clear that the systems of Fig. 32a and 32b are equivalent. For $\underline{G} = \underline{H} + \underline{K}$, then, we have

$$g_n(p_1T, \ldots, p_nT) = h_n(p_1T, \ldots, p_nT) = k_n(p_1T, \ldots, p_nT) \tag{122}$$

$$G_n(z_1, \ldots, z_n) = H_n(z_1, \ldots, z_n) + K_n(z_1, \ldots, z_n) \tag{123}$$

for $n = 0, 1, \ldots, N_g$ and $N_g = \max(N_h, N_k)$.

For the product of $\underline{H}$ and $\underline{K}$, the situation is as shown in Fig. 33. The hold circuits are needed in Fig. 33a, since the multiplier is a nonlinear no-memory device as discussed in Section III. The systems of Fig. 33a and 33b are clearly equivalent. For $\underline{G} = \underline{H} \cdot \underline{K}$, then, we have

$$g_n(p_1T, \ldots, p_nT) = \sum h_r(p_1T, \ldots, p_rT)\, k_q(p_{r+1}T, \ldots, p_nT) \tag{124}$$

$$G_n(z_1, \ldots, z_n) = \sum H_r(z_1, \ldots, z_r)\, K_q(z_{r+1}, \ldots, z_n), \tag{125}$$

where the sum in both (124) and (125) is taken over all pairs of integers $r \epsilon \{0, 1, \ldots, N_h\}$ and $q \epsilon \{0, 1, \ldots, N_k\}$, and both expressions hold for $n = 0, 1, \ldots, N_g$ with $N_g = N_h N_k$.

The cascade combination of $\underline{H}$ following $\underline{K}$ is shown in Fig. 34. It is clear that the systems of Fig. 34a and Fig. 34b are equivalent, while that of Fig. 34c is not equivalent to the others. Following the procedure used by Brilliant[26] for the continuous case, we have for the systems of Fig. 34a and 34b

$$g_n(p_1T, \ldots, p_nT) = \sum_{i=0}^{N_h} \sum \sum_{r_1=-\infty}^{\infty} \cdots \sum_{r_1=-\infty}^{\infty} h_i(r_1T, \ldots, r_iT) \prod_{j=1}^{i} k_{m_j}(\ldots, p_{()}T - r_jT, \ldots). \tag{126}$$

The numbers m_j are formed by taking all permutations of i non-negative integers which sum to n. The order of the subscripts on the $p_{()}$ in the brackets is not important. The second summation, for which no index is shown, is taken over all permutations of i numbers m_j whose sum is n. The corresponding kernel transforms are given by

$$G_n(z_1, \ldots, z_n) = \sum_{i=1}^{N_h} \sum H_i(\sigma_1, \ldots, \sigma_n) \prod_{j=1}^{i} K_{m_j}(\ldots, z_{()}, \ldots), \tag{127}$$

where σ_j is the product of arguments of K_{m_j}. As in (126), the second summation is over

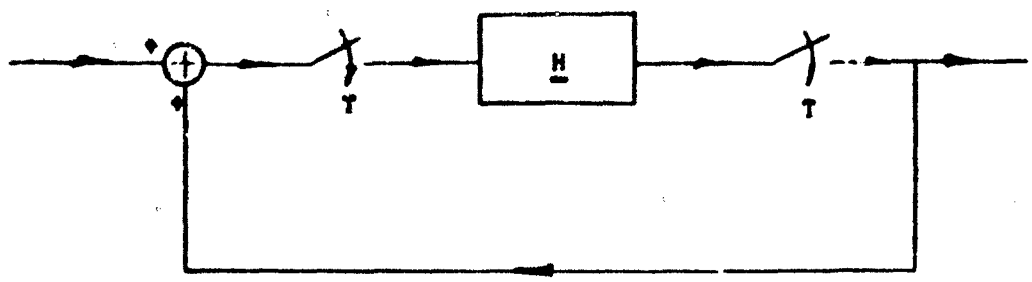


Fig. 35. Sampled-data feedback system.

Table 2. Kernel transforms for the feedback structure of Fig. 35.

$$G_1(z_1) = \frac{H_1(z_1)}{1-H_1(z_1)}$$

$$G_2(z_1,z_2) = \frac{H_2(z_1,z_2)}{[1-H_1(z_1z_2)][1-H_1(z_1)][1-H_1(z_2)]}$$

$$G_3(z_1,z_2,z_3) = \frac{1}{1-H_1(z_1z_2z_3)} \cdot \frac{1}{\prod_{i=1}^{3}[1-H_1(z_i)]} \cdot \left[H_3(z_1,z_2,z_3) + 2\,\frac{H_2(z_1,z_1z_2)H_2(z_2,z_3)}{1-H_1(z_2z_3)}\right]$$

all permutations of i numbers m_j whose sum is n, and the order of the $z_{()}$ is not important.

For the values n = 0, 1, and 2, assuming $h_o = k_o = 0$, we have from (127)

$$G_o = 0$$

$$G_1(z_1) = H_1(z_1)\,K(z_1) \tag{128}$$

$$G_2(z_1,z_2) = H_1(z_1z_2)\,K_2(z_1,z_2) + H_2(z_1,z_2)\,K_1(z_1)\,K_1(z_2).$$

The same relations given in Section II for a continuous nonlinear system in the feedback configuration of Fig. 12 hold for the sampled-data feedback system of Fig. 35, except that terms involving sums of s_i now involve products of z_i. Table 2 gives the z-transforms of the first few kernels of the sampled-data feedback system $\underline{G}$ in Fig. 35 in terms of the kernel transforms of the open-loop system $\underline{H}$. We emphasize that, as in Section II, these relations are formal; they assume that the feedback structure may be represented by a Volterra functional series.

Example 8

Consider the system of Fig. 36. The input and the output are sampled with the sampling interval T; x(t), $x^*(t)$, y(t), and $y^*(t)$ are the input, sampled input, output, and

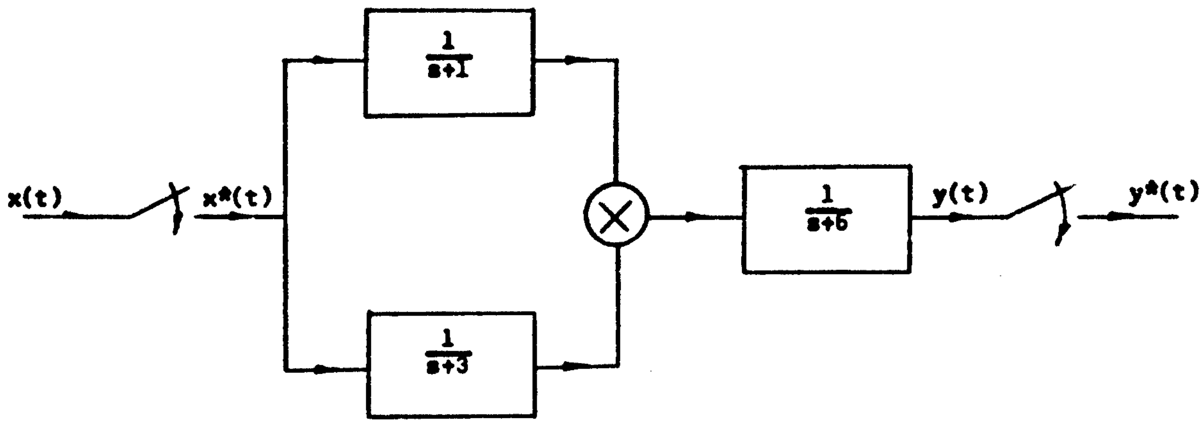


Fig. 36. Nonlinear sampled-data system of Example 8.

sampled output, respectively. The linear systems N_1, N_2, and N_3 have the system functions shown.

The z-transform of the kernel of this system was computed in Example 5 by using the definition of the z-transform, and again in Example 7 by using the frequency-domain technique of (102). It is given by

$$H_2(z_1, z_2) = \frac{\left(e^{-4T} - e^{-6T}\right) z_1^{-1} z_2^{-1}}{\left(1 - e^{-T} z_1^{-1}\right)\left(1 - e^{-3T} z_2^{-1}\right)\left(1 - e^{-6T} z_1^{-1} z_2^{-1}\right)}. \tag{129}$$

If $x(t) = e^{-2t} u_{-1}(t)$, then we have for the auxiliary output $y_{(2)}(t_1, t_2)$, according to Eq. 114

$$Y_{(2)}(z_1,z_2) = H_2(z_1,z_2)X(z_1)X(z_2)$$

$$= \frac{(e^{-4T}-e^{-6T})z_1^{-1}z_2^{-1}}{(1-e^{-6T}z_1^{-1}z_2^{-1})(1-e^{-T}z_1^{-1})(1-e^{-3T}z_2^{-1})(1-e^{-2T}z_1^{-1})(1-e^{-2T}z_2^{-1})}$$

$$= \frac{(e^{-4T}-e^{-6T})}{(e^{-T}-e^{-2T})(e^{-3T}-e^{-2T})} \cdot \frac{z_1^{-1}z_2^{-1}}{(1-e^{-6T}z_1^{-1}z_2^{-1})} \cdot$$

$$\left[\frac{e^{-T}}{1-e^{-T}z_1^{-1}} - \frac{e^{-2T}}{1-e^{-2T}z_1^{-1}}\right] \cdot \left[\frac{e^{-3T}}{1-e^{-3T}z_2^{-1}} - \frac{e^{-2T}}{1-e^{-2T}z_2^{-1}}\right] \tag{130}$$

Then multiplying out the terms in brackets and using the inspection technique for the association of variables we have

$$Y(z) = \frac{(e^{-4T}-e^{-6T})}{(e^{-T}-e^{-2T})(e^{-3T}-e^{-2T})} \cdot \frac{z^{-1}}{1-e^{-6T}z^{-1}}$$

$$\cdot \left[\frac{e^{-4T}}{1-e^{-4T}z^{-1}} - \frac{e^{-5T}}{1-e^{-5T}z^{-1}} - \frac{e^{-3T}}{1-e^{-3T}z^{-1}} + \frac{e^{-4T}}{1-e^{-4T}z^{-1}}\right] \tag{131}$$

from which the sample values of y(t), y(kT), can be easily computed.

VI. SYNTHESIS OF SAMPLED-DATA SYSTEMS

We shall now consider methods of synthesis for sampled-data systems. As in Section II, we consider nonlinear systems that can be represented by a single term from a Volterra functional series, and hence are characterized by a single Volterra kernel, $h_n(\tau_1, \dots, \tau_n)$. We assume that the sample values of the kernel, $h_n(k_1T, \dots, k_nT)$, and the corresponding z-transform, $H_n(z_1, \dots, z_n)$, are given. The basic building block for the synthesis of sampled-data systems is the digital computer; we consider methods of computing the output sample values $y(pT)$ from the input sample values $x(kT)$. We are concerned only with computation algorithms, and ignore both quantization and round-off error.

First, we consider direct computation of the convolution sum, then we discuss computation of the sample values of the auxiliary output function $y(p_1T, \dots, p_nT)$ through the associated partial difference equation, and finally we describe the decomposition of an n^{th}-degree system into a combination of linear sampled-data systems.

6.1 DIRECT COMPUTATION OF THE CONVOLUTION SUM

For an n^{th}-degree system with sampled input and output, the input-output relation was found in Section III to be

$$y(pT_o) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} x(k_1T_i) \dots x(k_nT_i)\, h_n(pT_o-k_1T_i, \dots, pT_o-k_nT_i). \quad (132)$$

If we assume that the kernel is realizable or causal and that the input is applied at $t=0$, (132) becomes

$$y(pT_o) = \sum_{k_n=0}^{a} \cdots \sum_{k_1=0}^{a} x(k_1T_i) \dots x(k_nT_i)\, h_n(pT_o-k_1T_i, \dots, pT_o-k_nT_i), \quad (133)$$

where $a = [p(T_o/T_i)]$ is the greatest integer function in $p(T_o/T_i)$. There are a^n terms in this sum; evaluation of the sum requires the computation and addition of each of these terms. This is a formidable task indeed if p is very large.

If we assume that the kernel is symmetrical and that only a finite number of the samples of the kernel are nonzero, that is, that the system has a finite memory, computation of the sum in (133) is simplified. But the amount of computation required will even then be prohibitive except in simple cases. Direct computation from the convolution sum, except in simple cases, is a severe problem even for linear systems; in the nonlinear case, the computational difficulty grows roughly exponentially with the degree of the system.

6.2 COMPUTATION FROM THE ASSOCIATED PARTIAL DIFFERENCE EQUATION

The input-output relation for the transforms of the input and output functions was found in Section V to be

$$Y_{(n)}(z_1, \ldots, z_n) = H_n(z_1, \ldots, z_n)\, X(z_1) \ldots X(z_n), \tag{134}$$

where $Y_{(n)}(z_1, \ldots, z_n)$ is the z-transform of the auxiliary output function $y_{(n)}(p_1T, \ldots, p_nT)$, $H_n(z_1, \ldots, z_n)$ is the z-transform of the kernel $h_n(k_1T, \ldots, k_nT)$, and $X(z)$ is the z-transform of the input $x(kT)$, and we assume that the input and output sampling intervals are equal and equal to T.

If $H_n(z_1, \ldots, z_n)$ can be expressed as a ratio of polynomials in $z_1^{-1}, \ldots, z_n^{-1}$,

$$H_n(z_1, \ldots, z_n) = \frac{P\left(z_1^{-1}, \ldots, z_n^{-1}\right)}{Q\left(z_1^{-1}, \ldots, z_n^{-1}\right)}, \tag{135}$$

where $P(\cdot)$ and $Q(\cdot)$ are polynomials, then we may write

$$Q\left(z_1^{-1}, \ldots, z_n^{-1}\right) Y_{(n)}(z_1, \ldots, z_n) = P\left(z_1^{-1}, \ldots, z_n^{-1}\right) X(z_1) \ldots X(z_n). \tag{136}$$

Recognizing from property 5.a.1 that $z_i^{-b_i}$ can be interpreted as a delay operator, we may take the inverse z-transform of both sides of (136) to find a linear partial difference equation with constant coefficients, which relates the auxiliary function $y_{(n)}(p_1T, \ldots, p_nT)$ to the input $x(kT)$. This partial difference equation provides a recursive computation algorithm for the auxiliary output function and hence for the output $y(pT)$. We illustrate this method with the following example.

<u>Example 9</u>

Let

$$H_2(z_1, z_2) = \frac{(e^{-4T} - e^{-6T})\, z_1^{-1} z_2^{-1}}{\left(1 - e^{-6T} z_1^{-1} z_2^{-1}\right)\left(1 - e^{-T} z_1^{-1}\right)\left(1 - e^{-3T} z_2^{-1}\right)}. \tag{137}$$

Then we have

$$\left[1 - e^{-T} z_1^{-1} - e^{-3T} z_2^{-1} + (e^{-4T} - e^{-6T}) z_1^{-1} z_2^{-1} + e^{-7T} z_1^{-2} z_2^{-1} + e^{-9T} z_1^{-1} z_2^{-2} - e^{-10T} z_1^{-2} z_2^{-2}\right] Y_{(2)}(z_1, z_2)$$

$$= (e^{-4T} - e^{-6T})\, z_1^{-1} z_2^{-1}\, X(z_1)\, X(z_2) \tag{138}$$

and hence

$$y(p_1T,p_2T) = e^{-T}y[(p_1-1)T,p_2T] + e^{-3T}y[p_1T,(p_2-1)T] - (e^{-4T}-e^{-6T})\, y[(p_1-1)T,(p_2-1)T]$$

$$- e^{-7T}y[(p_1-2)T,(p_2-1)T] - e^{-9T}y[(p_1-1)T,(p_2-2)T]$$

$$+ e^{-10T}y[(p_1-2)T,(p_2-2)T] + (e^{-4T}-e^{-6T})\, x[(p_1-1)T]\, x[(p_2-1)T],$$

in which we have dropped the subscript on $y(p_1T,p_2T)$ for convenience.

If we assume that $x(0) = 1$, and that all other input samples are zero, then (139) is a recursive relation for the computation of the inverse z-transform of $H_2(z_1,z_2)$, and thus we have arrived at another method of inversion for z-transforms, in addition to those given in Section V. For arbitrary $x(kT)$, in fact even for random $x(kT)$, since over any observation interval of arbitrary but finite length we could conceptually find the corresponding z-transform, we have a recursive computation algorithm for finding the output samples for a given sequence of input samples.

The major drawback of this method is that we must actually compute more than just the output samples of interest; we must compute the samples of the entire auxiliary function. Thus in Example 9, we must compute 2p samples, which are not of interest as far as the output $y(pT)$ is concerned, in order to compute the p^{th} output sample from the $(p-1)^{th}$ output sample.

This is in contrast to the situation for the linear case, in which the corresponding recursive relation is an ordinary difference equation, and all output samples that must be computed are actual output samples. In the nonlinear case, we must compute at each step the value of samples that are not of interest in order to get to the next sample of interest.

6.3 DECOMPOSITION INTO LINEAR SAMPLED-DATA SYSTEMS

In Section II we developed a procedure for determining whether or not an n^{th}-degree kernel can be realized exactly by using a finite number of linear systems and multipliers. By using the same trees and the rules for the interconnection of sampled-data systems which were given in Section V in place of the corresponding rules for the interconnection of continuous systems, we can determine from the kernel transform $H_n(z_1,\ldots,z_n)$ whether or not an n^{th}-degree sampled-data system can be decomposed into an interconnection of linear sampled-data systems. When such a decomposition is possible, we may form a computation algorithm for each of the component linear systems considered separately, and thus achieve a composite algorithm for the computation of the output samples of the nonlinear system from the input samples.

Example 10

Consider again the kernel transform of Example 9. We can write

$$H_2(z_1, z_2) = (e^{-4T} - e^{-6T}) \frac{z_1^{-1}}{1 - e^{-T} z_1^{-1}} \frac{z_2^{-1}}{1 - e^{-3T} z_2^{-1}} \frac{1}{1 - e^{-6T} (z_1 z_2)^{-1}}, \tag{140}$$

or as

$$H_2(z_1, z_2) = (e^{-4T} - e^{-6T}) K_a(z_1) K_b(z_2) K_c(z_1 z_2), \tag{141}$$

where $K_a(z)$, $K_b(z)$, and $K_c(z)$ are the systems functions of linear sampled-data systems.

The equations

$$\begin{aligned} u[pT] &= e^{-T} u[(p-1)T] + x[(p-1)T] \\ v[pT] &= e^{-3T} v[(p-1)T] + x[(p-1)T] \\ r(pT) &= u(pT)\, v(pT) \\ y[pT] &= e^{-6T} y[(p-1)T] + e^{-4T} - e^{-6T})\, r[pT] \end{aligned} \tag{142}$$

then form a computation algorithm for computation of the output samples $y(pT)$ from the input samples $x(kT)$.

Example 11

Consider again the kernel of Example 3.

$$h_2(\tau_1, \tau_2) = (1-\tau_1-\tau_2)\, u_{-1}(1-\tau_1-\tau_2)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) \tag{143}$$

We found in Example 3 that, as a continuous system, this kernel was not realizable exactly with a finite number of linear systems and multipliers. It is of interest to examine the corresponding sampled kernel. The z-transform of the kernel (143), if we assume a sampling interval of T, is

$$H_2(z_1, z_2) = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} (1-k_1T-k_2T)\, u_{-1}(1-k_1T-k_2T)\, u_{-1}(k_1T)\, u_{-1}(k_2T)\, z_1^{-k_1} z_2^{-k_2}. \tag{144}$$

Make the change of index $k_1 + k_2 = k$. Then (144) becomes

$$H_2(z_1, z_2) = \sum_{k=0}^{1/T} \sum_{k_1=0}^{k} (1-kT) z_1^{-k_1} z_2^{-(k-k_1)} = \sum_{k=0}^{\infty} (1-kT) z_2^{-k} \sum_{k_1=0}^{k} (z_1/z_2)^{-k_1}. \tag{145}$$

Now

$$\sum_{k_1=0}^{k} (z_1/z_2)^{-k_1} = \frac{1 - (z_1/z_2)^{-k-1}}{1 - (z_1/z_2)^{-1}}. \tag{146}$$

The remaining sum may be interpreted as the sum of a step, a negative ramp, and a delayed positive ramp.

$$\sum_{k=0}^{1/T} (1-kT)z^{-k} = \frac{1}{1-z^{-1}} - T\frac{z^{-1}}{(1-z^{-1})^2} + Tz^{-1/T}\frac{z^{-1}}{(1-z^{-1})^2}$$

$$= \frac{1-(1+T)z^{-1} + Tz^{-(1/T)-1}}{(1-z^{-1})^2} = F(z). \tag{147}$$

Combining these results, we evaluate (145) as

$$H_2(z_1, z_2) = \frac{1}{1-(z_1/z_2)^{-1}}\left[F(z_2) - (z_1/z_2)^{-1}F(z_1)\right], \tag{148}$$

or

$$H_2(z_1,z_2) = \frac{\left(1-z_1^{-1}\right)^2\left(z_2^{-1}-(1+T)z_2^{-2}+Tz_2^{-(1/T)-2}\right) - \left(1-z_2^{-1}\right)^2\left(z_1^{-1}-(1+T)z_1^{-2}+Tz_1^{-(1/T)-2}\right)}{\left(z_2^{-1}-z_1^{-1}\right)\left(1-z_1^{-1}\right)^2\left(1-z_2^{-1}\right)^2}. \tag{149}$$

The factor $\left(z_2^{-1}-z_1^{-1}\right)$ in the denominator may be divided out to give

$$H_2(z_1, z_2)$$

$$= \frac{1-(1+T)\left(z_1^{-1}+z_2^{-1}\right) + (1+2T)z_1^{-1}z_2^{-1} + T\left(z_2^{-1}z_1^{-(1/T)}+z_1^{-(1/T)-1}-2z_1^{-(1/T)-1}z_2^{-1}\right)}{\left(1-z_1^{-1}\right)^2\left(1-z_2^{-1}\right)^2}$$

$$+ \frac{T\left(z_2^{-(1/T)-1}+z_2^{-(1/T)}z_1^{-1}+\ldots+z_2^{-2}z_1^{-(1/T)+1}\right)}{\left(1-z_2^{-1}\right)^2}. \tag{150}$$

Examination of (150) shows that $H_2(z_1, z_2)$ can be represented as an interconnection of linear sampled-data systems, although this realization will be rather complex, particularly for small T.

VII. TIME-DOMAIN SYNTHESIS TECHNIQUE

7.1 IMPULSE-TRAIN TECHNIQUES IN LINEAR SYSTEM THEORY

An important feature of the use of the convolution or superposition integral in linear system theory is the possibility of impulse-train techniques.[27] The input-output relation for a linear time-invariant system may be given by the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau, \tag{151}$$

where x(t) is the system input, y(t) is the output, and $h(\tau)$ is the unit impulse response or kernel of the system. A generalized expression is

$$y^{(m+k)}(t) = \int_{-\infty}^{+\infty} h^{(m)}(\tau)\, x^{(k)}(t-\tau)\, d\tau, \tag{152}$$

where m and k are positive or negative integers, with $f^{(k)}(x)$ representing the k^{th} derivative of f(x) if k is positive and the k^{th} successive integration, as in (153), when k is negative.

$$f^{(-1)}(x) = \int_{-\infty}^{x} f(y)\, dy. \tag{153}$$

If some derivative of $h(\cdot)$ or $x(\cdot)$ yields only impulses, evaluation of (152) becomes very simple. This technique is useful in the evaluation of (151), in finding the transform of $h(\cdot)$, and in finding approximants to $h(\cdot)$, and is thus extremely useful in the synthesis of a linear system when $h(\cdot)$ is given.

7.2 GENERALIZATION OF IMPULSE-TRAIN TECHNIQUES TO NONLINEAR SYSTEMS

We shall now demonstrate the use of impulse-train techniques for nonlinear systems. Consider a nonlinear system for which the input-output relation

$$y(t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n(\tau_1, \ldots, \tau_n)\, x(t-\tau_1) \ldots x(t-\tau_n)\, d\tau_1 \ldots d\tau_n \tag{154}$$

applies, where x(t) is the input, y(t) the output, and $h_n(\tau_1, \ldots, \tau_n)$ the kernel of the system. This relation may be generalized just as in the linear case to give

$$y(t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n^{(m)}(\tau_1, \ldots, \tau_n)\, x^{(k)}(t-\tau_1) \ldots x^{(k)}(t-\tau_n)\, d\tau_1 \ldots d\tau_n. \tag{155}$$

Here the superscripts $x(\cdot)$ have the same significance as in (152), and m + k = 0.

We define

$$h_n^{(m)}(\tau_1, \dots, \tau_n) = \frac{\partial^{nm} h_n(\tau_1, \dots, \tau_n)}{\partial \tau_n^m \dots \partial \tau_1^m}$$

for m positive. For m = -1, we define

$$h_n^{(-1)}(\tau_1, \dots, \tau_n) = \int_{-\infty}^{\tau_n} \dots \int_{-\infty}^{\tau_1} h_n(\eta_1, \dots, \eta_n)\, d\eta_1 \dots d\eta_n. \tag{156}$$

For m any negative integer, $h_n^{(m)}(\tau_1, \dots, \tau_n)$ is found by repeated application of (156).

Although (155) is true for any n, it appears to be most useful for n = 2, since we may often use graphical techniques in this case. Examples of the use of (155) for calculation of kernel transforms and synthesis of a given kernel for second-degree systems are given below.

7.3 EXAMPLES OF THE USE OF IMPULSE-TRAIN TECHNIQUES FOR SECOND-DEGREE SYSTEMS

Example 12

Consider the second-degree kernel given by

$$h_2(\tau_1, \tau_2) = u_{-1}(\tau_1)\, u_{-1}(1-\tau_1)\, u_{-1}(\tau_2)\, u_{-1}(1-\tau_2) \tag{157}$$

and sketched in Fig. 37a. Form the partial derivatives with respect to τ_1 and then with respect to τ_2. This may be accomplished either graphically or analytically, with the following result:

$$\begin{aligned} \frac{\partial h_2}{\partial \tau_1} &= [u_o(\tau_1) - u_o(1-\tau_1)]\, u_{-1}(\tau_2)\, u_{-1}(1-\tau_2) \\ \frac{\partial^2 h_2}{\partial \tau_2 \partial \tau_1} &= u_o(\tau_1)\, u_o(\tau_2) - u_o(\tau_1)\, u_o(1-\tau_2) + u_o(1-\tau_1)\, u_o(1-\tau_2) - u_o(1-\tau_1)\, u_o(\tau_2). \end{aligned} \tag{158}$$

These partials are sketched in Fig. 37b and 37c. A type of singularity which we shall call an "impulsive fence" occurs in the partial with respect to τ_1 (Fig. 37b).

The four impulses of the second partial can be realized as shown in Fig. 38a, and combined as a sum to give the system of Fig. 38b, which is a realization of the second partial. Simplification yields the equivalent system shown in Fig. 39. Precascading an ideal integrator, as in Fig. 40, yields a system which realizes the original kernel. Note that only one integrator is required, although two differentiations were performed. Simplification yields the system of Fig. 41. Of course, in this simple example, we could have found the system of Fig. 41 directly from the expression for the kernel (157).

We may also use this technique to obtain the kernel transform, as in Eqs. 159 and 160,

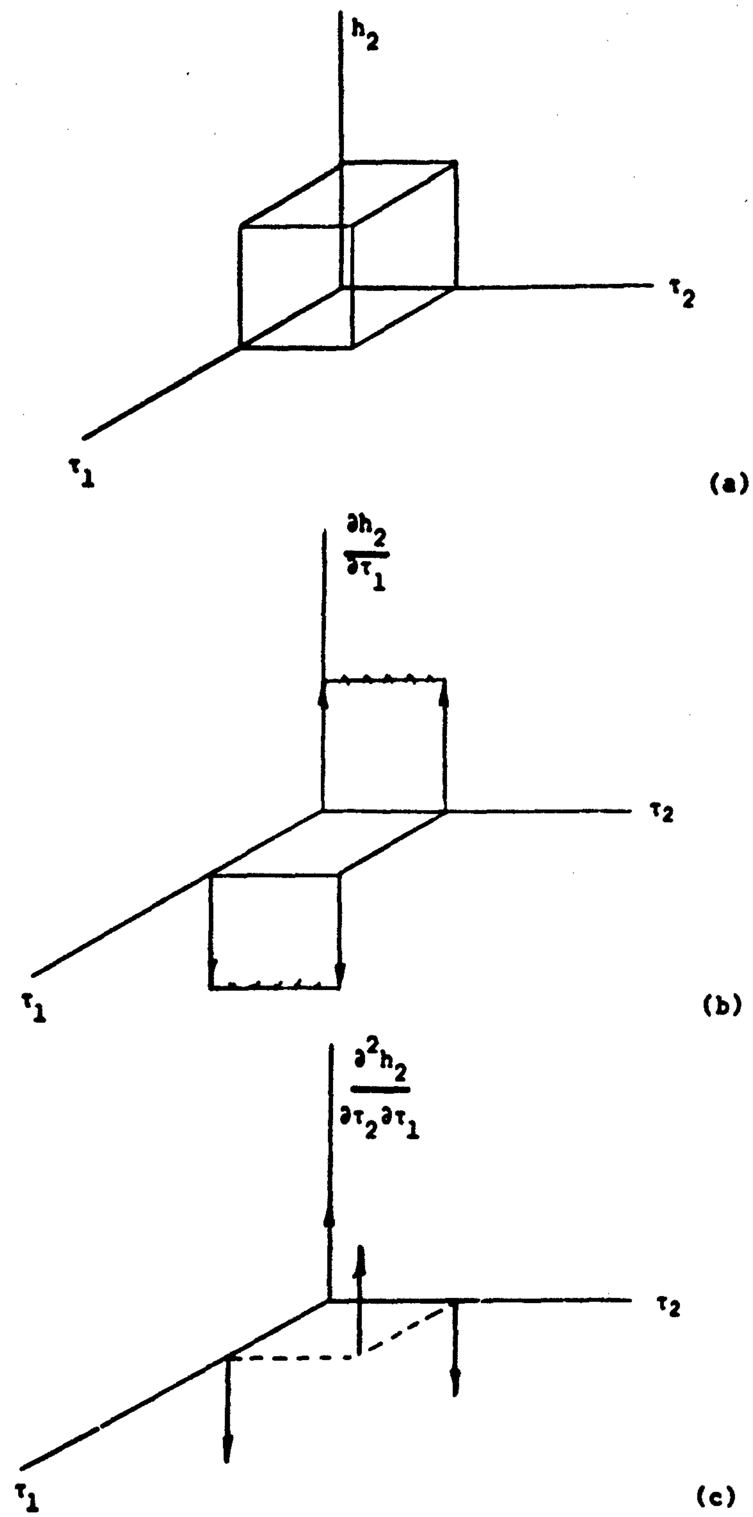


Fig. 37. The kernel of Example 12.

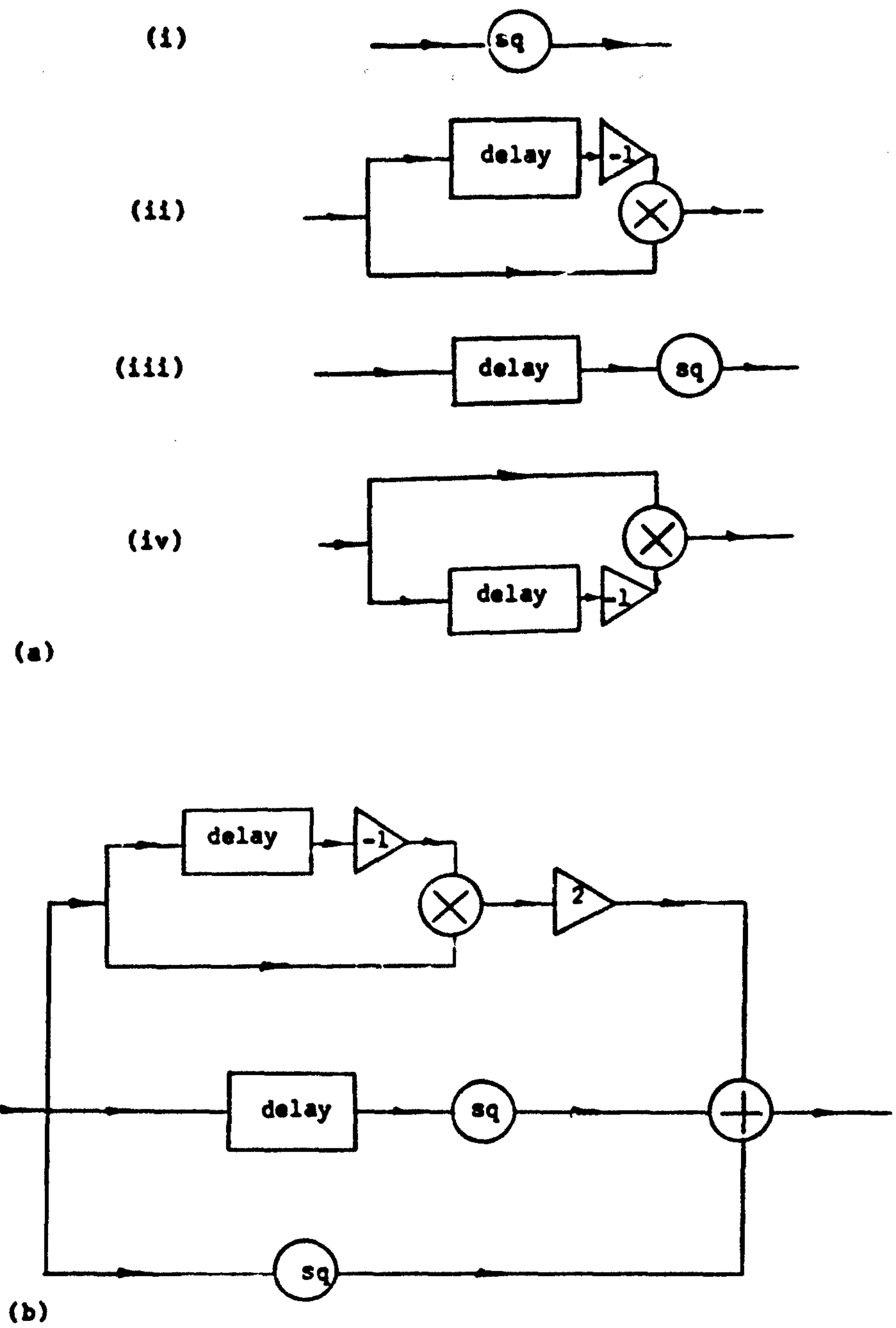


Fig. 38. Realization of $\frac{\partial^2 h_2}{\partial\tau_1 \partial\tau_2}$ of Example 12.

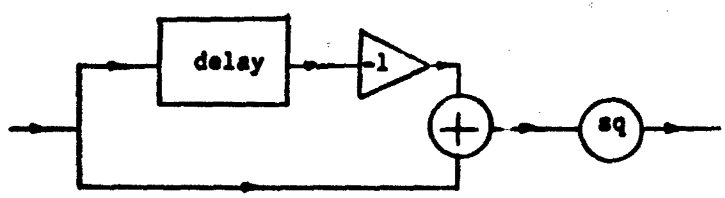


Fig. 39. Simplified realization of $\frac{\partial^2 h_2}{\partial\tau_2 \partial\tau_1}$ of Example 12.

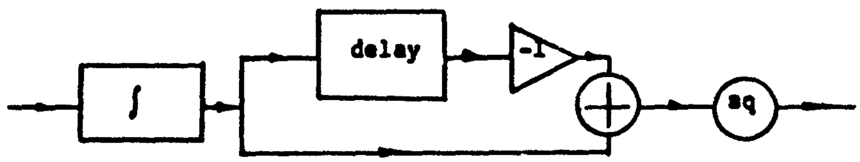


Fig. 40. Realization of the kernel $h_2(\tau_1, \tau_2)$ of Example 12.

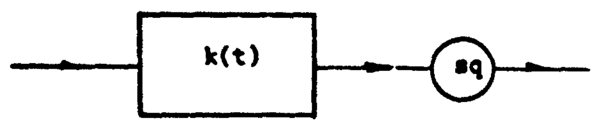


Fig. 41. Simplified realization of the kernel of Example 12, with $k(t) = u_{-1}(t)\, u_{-1}(1-t)$.

since the transforms of the components of the singular kernel of Fig. 37c can be written by inspection. Factoring the transform expression of (160) to separate variables, we obtain the transform expression (161). The system of Fig. 41 is recognizable from this form also.

$$\frac{\partial^2 h_2}{\partial\tau_2 \partial\tau_1} \longleftrightarrow 1 + e^{-s_1} e^{-s_2} - e^{-s_1} - e^{-s_2} \tag{159}$$

$$h_2 \longleftrightarrow \frac{1 + e^{-s_1} e^{-s_2} - e^{-s_1} - e^{-s_2}}{s_1 s_2} \tag{160}$$

$$H_2(s_1, s_2) = \left(\frac{1 - e^{-s_1}}{s_1}\right)\left(\frac{1 - e^{-s_2}}{s_2}\right). \tag{161}$$

<u>Example 13</u>

Consider the second-degree system characterized by

$$h_2(\tau_1, \tau_2) = (1-\tau_1-\tau_2)\, u_{-1}(1-\tau_1-\tau_2)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) \tag{162}$$

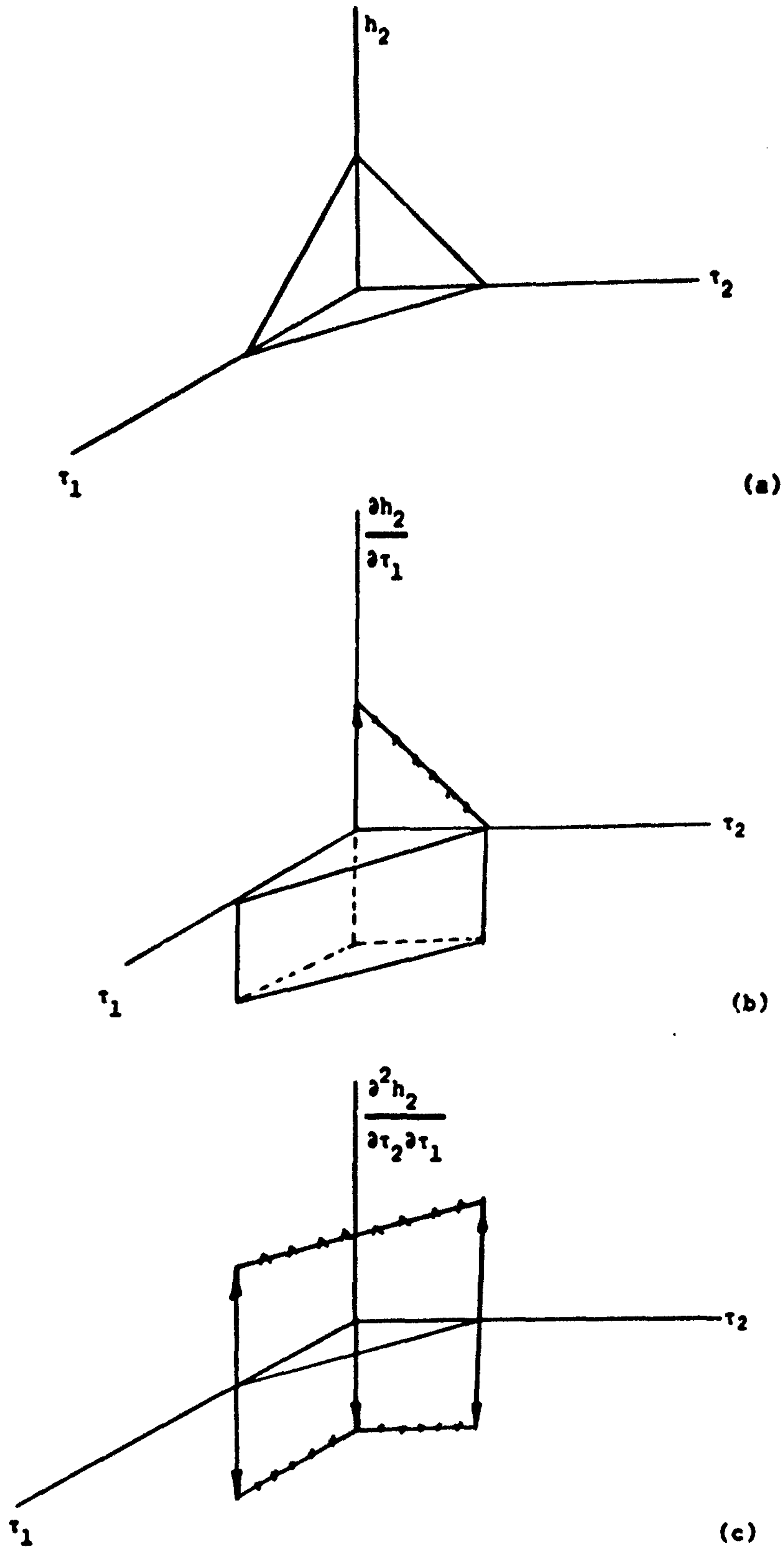


Fig. 42. The kernel of Example 13.

and shown in Fig. 42a. Partial differentiation of this kernel yields

$$\frac{\partial h_2}{\partial \tau_1} = - u_{-1}(1-\tau_1-\tau_2)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) + (1-\tau_2)\, u_{-1}(1-\tau_2)\, u_{-1}(\tau_2)\, u_o(\tau_1)$$

$$\begin{aligned}\frac{\partial^2 h_2}{\partial \tau_2 \partial \tau_1} &= u_o(1-\tau_1-\tau_2)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) - u_{-1}(1-\tau_1)\, u_{-1}(\tau_1)\, u_o(\tau_2) \\ &\quad - u_{-1}(1-\tau_2)\, u_{-1}(\tau_2)\, u_o(\tau_1) + u_o(\tau_1)\, u_o(\tau_2).\end{aligned} \tag{163}$$

These partial derivatives are sketched in Fig. 42b and 42c.

Let us find the transform of the kernel (162). We shall look at the derivative (163), taking each term separately and summing. The transform of the second partial derivative is thus

$$1 - \frac{1 - e^{-s_1}}{s_1} - \frac{1 - e^{-s_2}}{s_2} + \frac{e^{-s_1} - e^{-s_2}}{s_2 - s_1}.$$

Simplification yields

$$\frac{s_1^2\left(1 - s_2 - e^{-s_2}\right) - s_2^2\left(1 - s_1 - e^{-s_1}\right)}{s_1 s_2 (s_2 - s_1)}$$

as the transform of the second partial, and hence we have

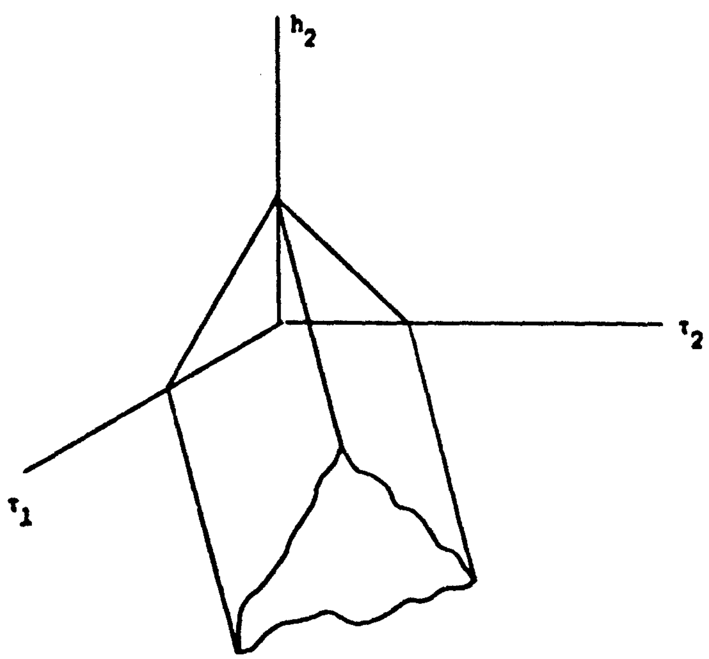


Fig. 43. The kernel of Example 14.

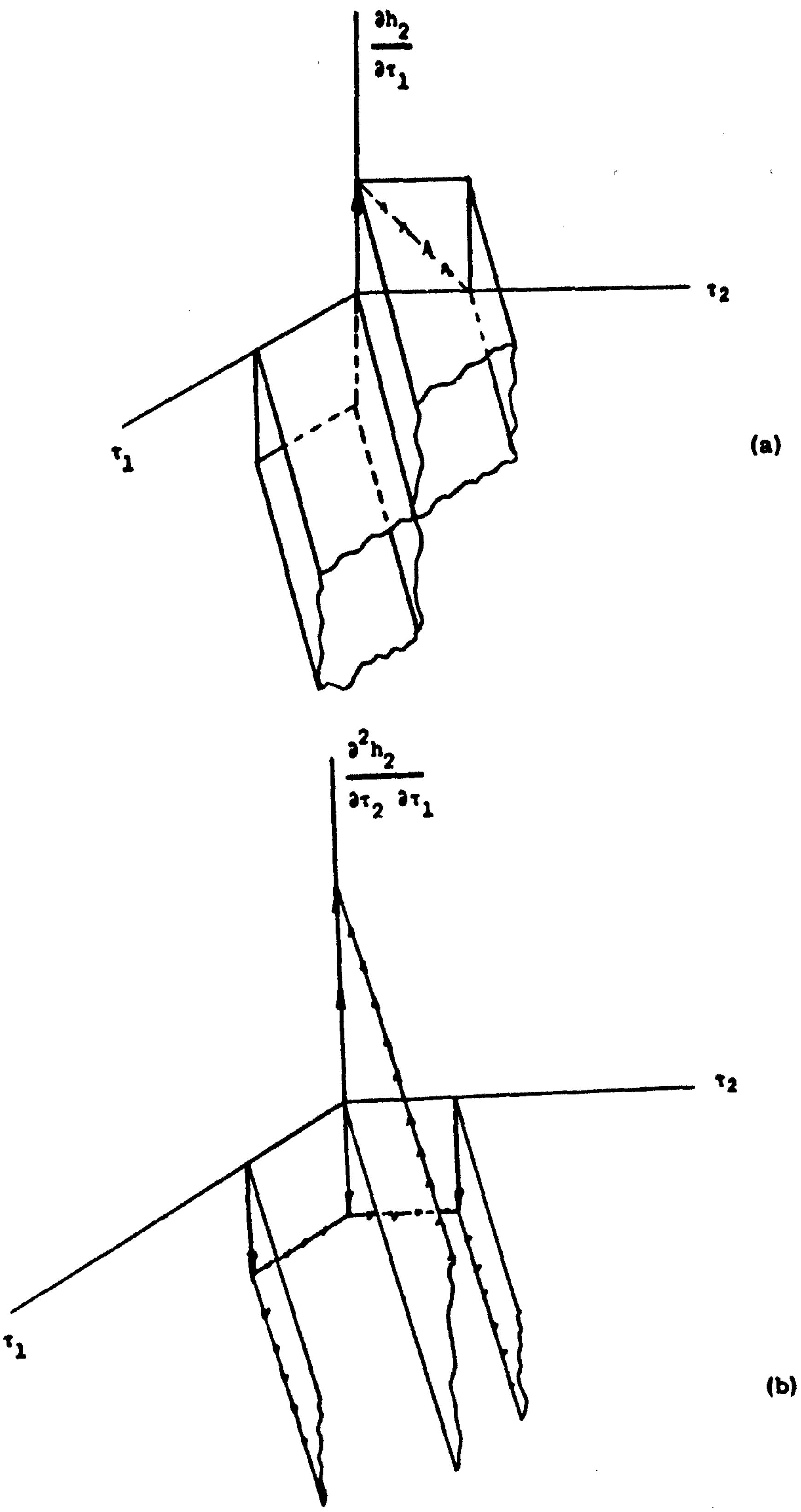


Fig. 44. The kernel of Example 14.

$$H_2(s_1, s_2) = \frac{s_1^2\left(1 - s_2 - e^{-s_2}\right) - s_2^2\left(1 - s_1 - e^{-s_1}\right)}{s_1^2 s_2^2 (s_2 - s_1)}. \tag{164}$$

<u>Example 14</u>

Consider the kernel given by

$$h_2(\tau_1, \tau_2) = (1 - |\tau_1 - \tau_2|)\, u_{-1}(1 - |\tau_1 - \tau_2|)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) \tag{165}$$

and shown in Fig. 43. This kernel may be differentiated as shown in Fig. 44a and 44b. The transform corresponding to Fig. 44b is

$$1 + \frac{2}{s_1 + s_2} - \frac{1 - e^{-s_1}}{s_1} - \frac{1 - e^{-s_2}}{s_2} - \frac{e^{-s_1}}{s_1 + s_2} - \frac{e^{-s_2}}{s_1 + s_2}. \tag{166}$$

Simplification and division by s_1 and s_2 yields the kernel transform for (165):

$$H_2(s_1, s_2) = \frac{-s_2^2\left(1 - s_1 - e^{-s_1}\right) - s_1^2\left(1 - s_2 - e^{-s_2}\right)}{s_1^2 s_2^2 (s_1 + s_2)}. \tag{167}$$

We see by inspection of this transform expression that the kernel is realizable by a

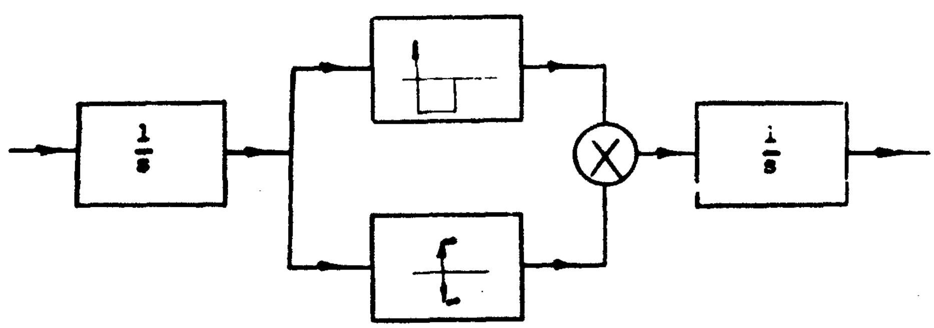


Fig. 45. Realization of the kernel of Example 14.

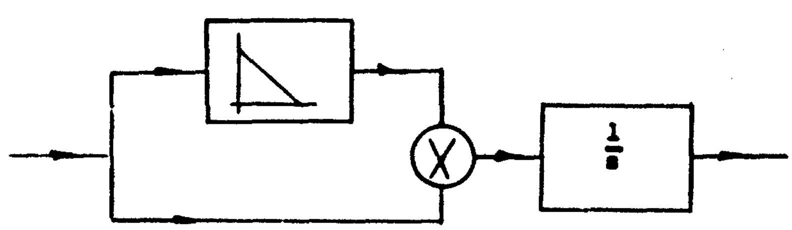


Fig. 46. Simplified realization of the kernel of Example 14.

finite number of linear systems and multipliers. An equivalent but unsymmetrical kernel transform is

$$H_2^*(s_1, s_2) = \frac{1}{s_1} \cdot \frac{1}{s_2} \cdot \frac{e^{-s_1} + s_1 - 1}{s_1} \cdot s_2 \cdot \frac{2}{s_1 + s_2}. \tag{168}$$

This kernel can be realized readily with only one multiplier as shown in Fig. 45 and in simplified form in Fig. 46.

7.4 REMARKS

Some important observations can be drawn from these examples. From the kernel transforms of these examples, we can see that the kernels of Examples 12 and 14 are of the class that can be realized exactly with a finite number of linear systems and multipliers, while the kernel of Example 13 (considered also in Example 3) cannot be realized with a finite number of linear systems and multipliers.

We might attempt to approximate an arbitrary kernel $h_2(\tau_1, \tau_2)$ with planes, so that we could differentiate with respect to τ_1 and τ_2 to obtain a new function consisting of impulses and impulsive fences; if we could find a system that realized this singular kernel, then the original kernel would be realized by this system cascaded after an ideal integrator. Manipulation of the resulting system as in Example 12 might lead to a quite simple realization.

7.5 REALIZATION OF IMPULSIVE FENCES

Exactly Realizable Impulsive Fences. Example 13 shows, however, that not all impulsive fences are realizable with a finite number of multipliers and linear systems. In fact, a little reflection shows that the only impulsive fences that can be realized with one

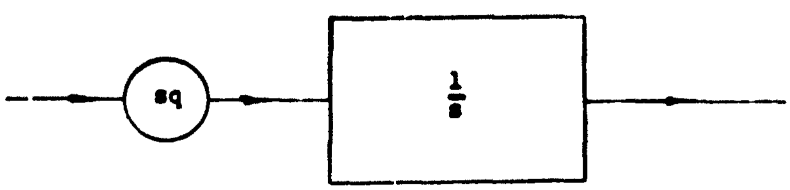


Fig. 47. A system whose kernel is an impulsive function.

multiplier and linear systems are those that lie along lines intersecting the τ_1 or τ_2 axes at a 45° angle, or along lines parallel to the axes. Such an impulsive fence is

$$f(\tau_1, \tau_2) = u_0(\tau_1 - \tau_2)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) \tag{169}$$

which has the transform

$$F(s_1, s_2) = \frac{1}{s_1 + s_2}. \tag{170}$$

This is realizable as shown in Fig. 47.

A unit impulsive fence passing through the origin of the τ_1, τ_2 plane at any other angle will not be realizable with a finite number of linear systems and multipliers. For example,

$$g(\tau_1, \tau_2) = u_o(\tau_1 - a\tau_2)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) \tag{171}$$

has the transform

$$G(s_1, s_2) = \frac{1}{s_2 + as_1}. \tag{172}$$

Approximation Realization of Impulsive Fences. The impulsive fence

$$u_o(1-\tau_1-\tau_2)\, u_{-1}(\tau_1)\, u_{-1}(\tau_2) \longleftrightarrow \frac{e^{-s_1} - e^{-s_2}}{s_2 - s_1} \tag{173}$$

of Example 13 lies perpendicular to the 45° lines and thus cannot be realized exactly with linear systems and multipliers. We can, however, realize this impulsive fence approximately by means of an appropriately weighted set of isolated impulses occurring on the same line in the τ_1, τ_2 plane.

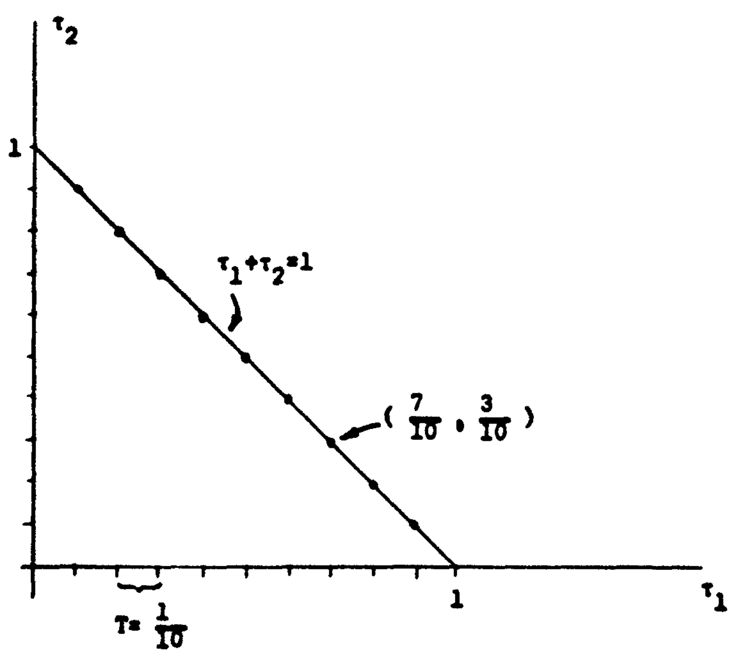


Fig. 48. Approximation of an impulsive fence.

Partition the line $\tau_1 + \tau_2 = 1$ into intervals of length T as shown in Fig. 48, so that the end points of the intervals fall at the points $(kT, 1-kT)$, $k = 0, \ldots, 1/T$. At each of these points we place an impulse whose area is equal to T times the amplitude of the envelope of the impulsive fence at that point. Hence we have

$$\sum_{k=0}^{1/T} T u_o(\tau_1 - kT)\, u_o(\tau_2 - 1 + kT) \tag{174}$$

as the proposed approximation to the impulsive fence of (173). The transform of (174) is

$$\sum_{k=0}^{1/T} T\, e^{-kTs_1} e^{-(1-kT)s_2} = T\, e^{-s_2} \sum_{k=0}^{1/T} e^{-kT(s_1-s_2)}. \tag{175}$$

We can write the expression on the right in (175) in closed form as

$$T\, e^{-s_2} \frac{1 - e^{-(s_1-s_2)(1+T)}}{1 - e^{-T(s_1-s_2)}} = e^{-s_2}\left(1 - e^{-(s_1-s_2)(1+T)}\right) \frac{T}{1 - e^{-T(s_1-s_2)}}. \tag{176}$$

As $T \to 0$, by l'Hôpital's Rule, we have the limit

$$\frac{e^{-s_2} - e^{-s_1}}{s_1 - s_2}. \tag{177}$$

Thus, to approximate this impulsive fence, we need only isolated inpulses along the line of the impulsive fence, weighted according to the envelope of the impulsive fence. Other impulsive fences may be approximated in exactly the same fashion.

VIII. ANCILLARY RESULTS

8.1 TIME-INVARIANT SYSTEMS AND TIME-VARIANT SYSTEMS

Physical situations can sometimes be modeled with either a time-variant system or a nonlinear system, according to the viewpoint one adopts. There is a very close connection between time-variant systems and nonlinear time-invariant systems, as we shall point out.

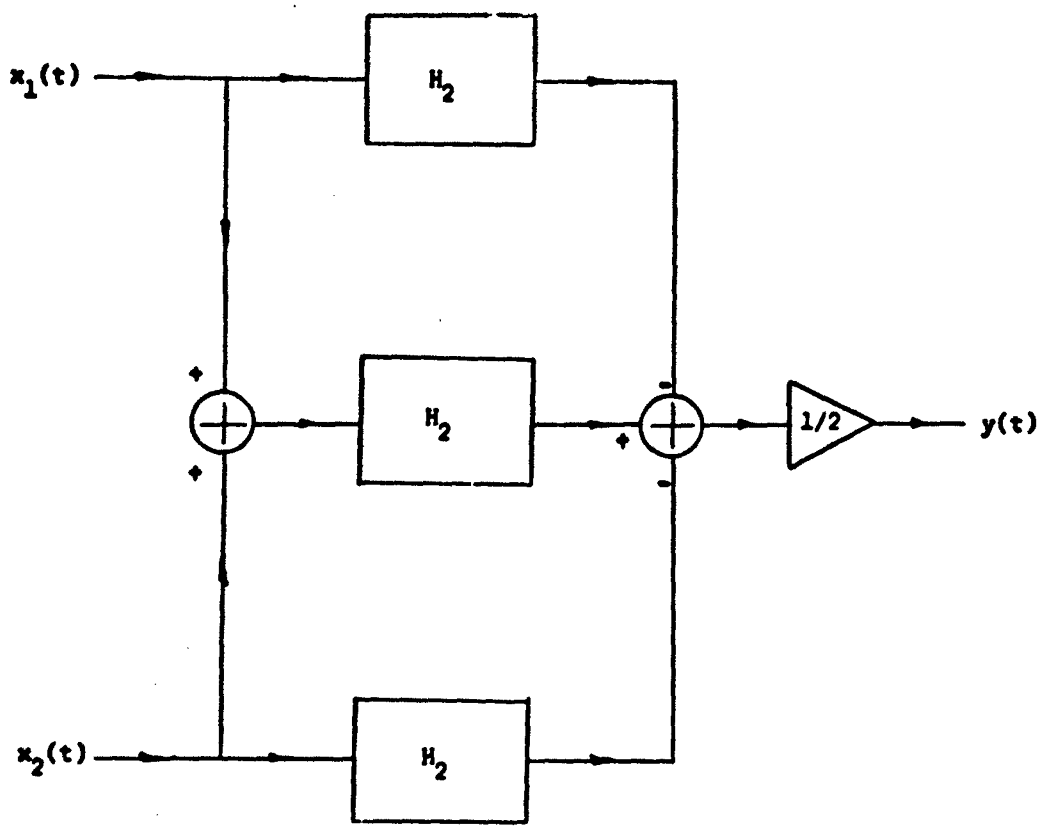


Fig. 49. Configuration for cross-term output.

Consider a second-degree system characterized by the symmetrical kernel $h_2(\tau_1, \tau_2)$, with input x(t) and output y(t). The input-output relationship is given by

$$y(t) = \iint h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2. \tag{178}$$

Consider the configuration of Fig. 49 The three systems are identical, and the inputs and outputs are combined as shown. After scaling the gain by a factor of one-half, we obtain the output

$$y(t) = \iint h_2(\tau_1, \tau_2)\, x_1(t-\tau_1)\, x_2(t-\tau_2)\, d\tau_1 d\tau_2. \tag{179}$$

Now suppose that we let $x_2(t) = u_o(t)$ and keep $x_1(t)$ arbitrary. The output now becomes

$$y(t) = \int h_2(\tau_1, t)\, x_1(t-\tau_1)\, d\tau_1. \tag{180}$$

That is, by application of a timing pulse $x_2(t)$, with second-degree time-invariant systems we have precisely the situation encountered in a linear time-variant system. The restriction as to the class of linear time-variant systems may be represented in this way or determined by the kernels $h_2(\tau_1, \tau_2)$ that we permit. We could also choose the input $x_2(t)$ in other ways, for example, an impulse train or other periodic signal; we may choose to make $x_2(t)$ random in order to model a randomly time-variant situation.

We have assumed in (180) that the kernel $h_2(\tau_1, \tau_2)$ is symmetrical; however, if we are able to identify in a realization of the second-degree system which portions are identified with τ_1 and which with τ_2, then the requirement of symmetry is not really necessary. For example, in Fig. 50, if the upper branch is identified with τ_1 and the lower branch with τ_2, we may apply $x_1(t)$ and $x_2(t)$ as shown to obtain a time-variant system without the restriction to a symmetrical kernel.

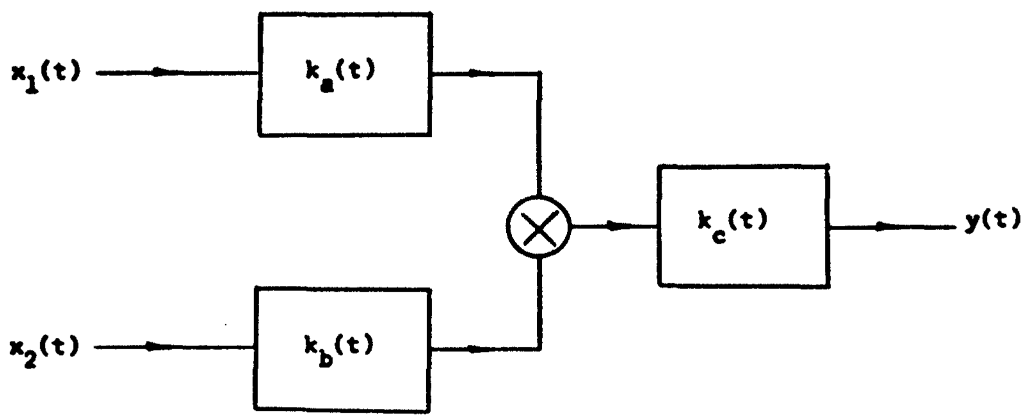


Fig. 50. A linear time-variant system.

Thus, if a time-variant kernel $h(\tau, t)$ is given, we can realize $h(\tau_1, \tau_2)$ as a second-degree nonlinear system using any of the properties or techniques of the preceding sections, but keeping track of which parts of the realization we wish to identify with τ_1 and which with τ_2; then application of a timing pulse or signal to the τ_2 branches and the input $x(t)$ to the τ_1 branches yields a realization of $h(\tau, t)$.

Second-degree time-variant systems may be obtained from third-degree time-invariant systems from the configuration shown in Fig. 51. The output is given by

$$v(t) = \iiint h_3(\tau_1, \tau_2, \tau_3)\, x_1(t-\tau_1)\, x_2(t-\tau_2)\, x_3(t-\tau_3)\, d\tau_1 d\tau_2 d\tau_3. \tag{181}$$

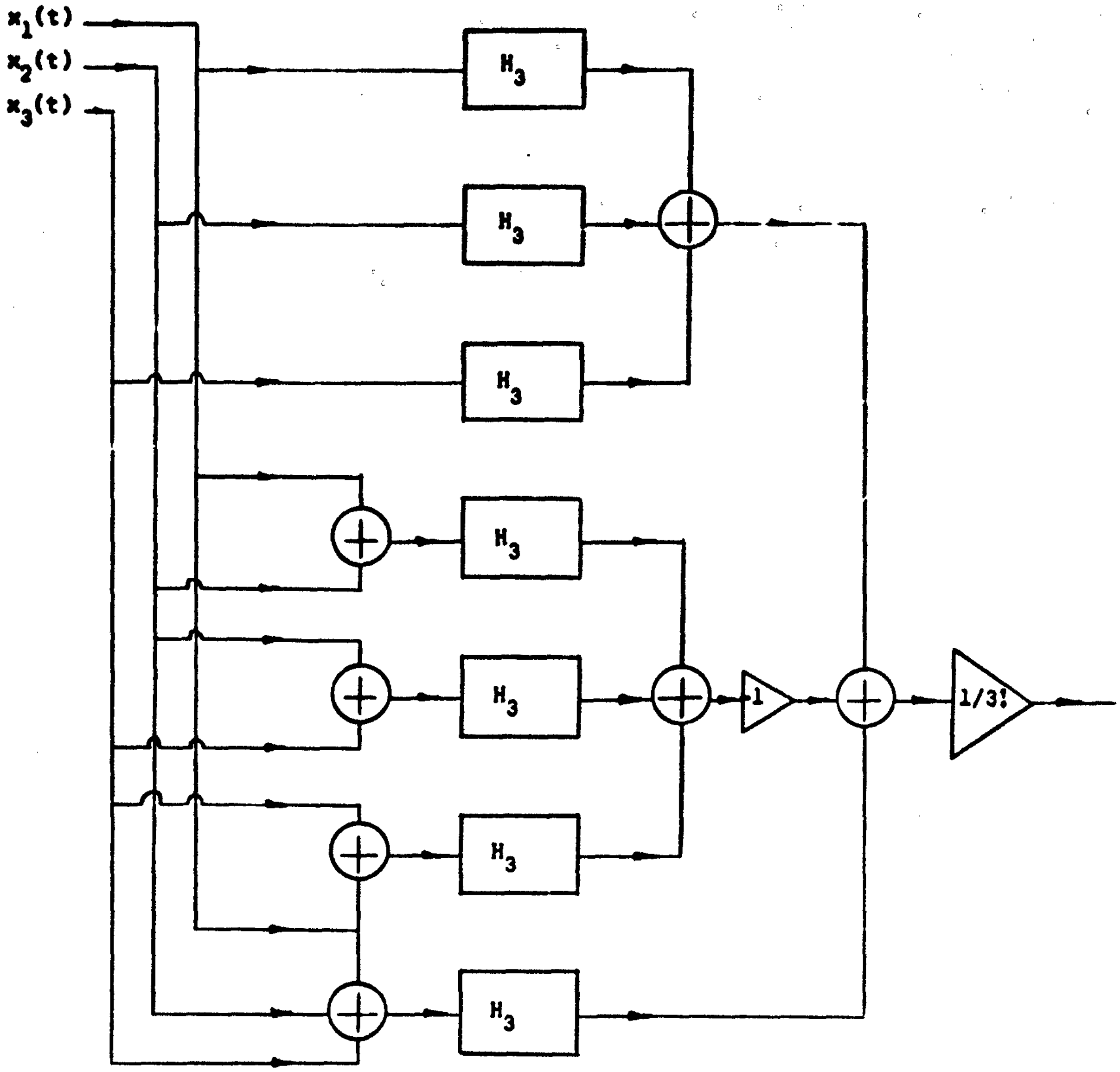


Fig. 51. Configuration for cross-term output.

If we allow $x_1(t) = x_2(t)$ to be arbitrary, but make $x_3(t) = u_0(t)$, we obtain

$$y(t) = \iint h_3(\tau_1, \tau_2, t)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2, \tag{182}$$

in which we have dropped the subscripts on the input x(t). This can be interpreted as representing a second-degree time-variant system. Other variations are also possible, and, with rapidly increasing complexity, we may consider higher degree systems too.

8.2 RELATION BETWEEN INTEGRAL AND DIFFERENTIAL CHARACTERIZATIONS OF NONLINEAR SYSTEMS

Consider the nonlinear system shown in block diagram form in Fig. 52. N_1, N_2, and N_3 are linear systems, and N_4 is a multiplier. The behavior of the system can be characterized by the set of Eqs. 183-186.

$$\frac{dz(t)}{dt} + az(t) = x(t) \tag{183}$$

$$\frac{dw(t)}{dt} + bw(t) = x(t) \tag{184}$$

$$\frac{d^2y(t)}{dt^2} + d\frac{dy(t)}{dt} + ey(t) = cr(t) + \frac{dr(t)}{dt} \tag{185}$$

$$r(t) = w(t)\, z(t), \tag{186}$$

where x(t) is the input, y(t) is the output, and w(t), z(t), and r(t) are the inputs and output of the multiplier, as shown in Fig. 52. Equations 183-186 describe the behavior of N_1 through N_4. We shall assume that all initial conditions are zero.

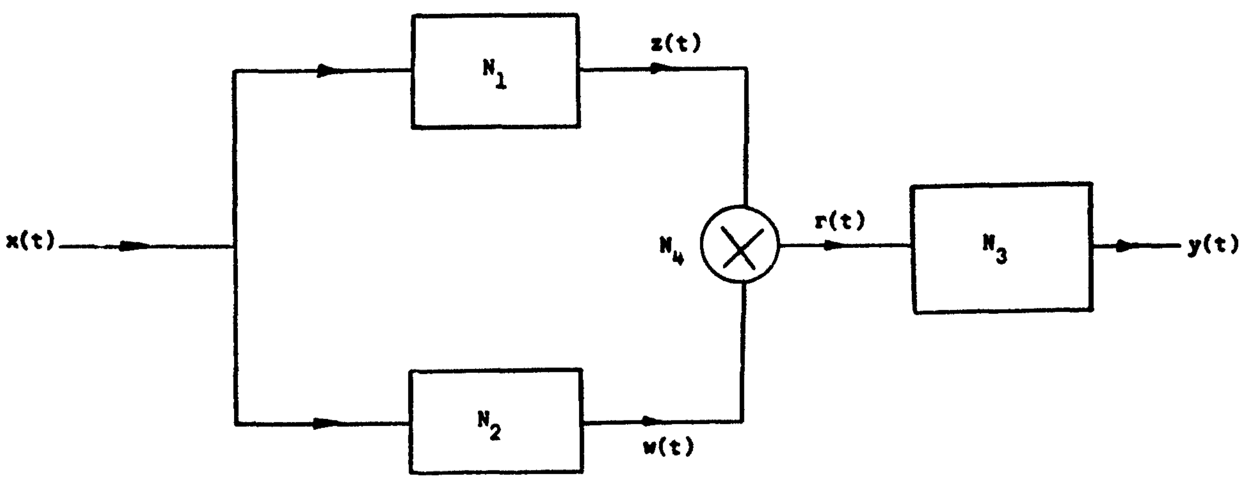


Fig. 52. A simple nonlinear system.

We would like to find a differential equation relating y(t) and x(t); that is, we would like to eliminate w(t), z(t), and r(t) in Eqs. 183-186. In order to do so, we shall extend the domain of definition from a line to a plane, and look along the 45° line in the plane.

Define $\hat{r}(t_1, t_2) = w(t_1)\, z(t_2)$ and $\hat{y}(t_1, t_2)$ such that y(t) is $\hat{y}(t_1, t_2)\big|_{t_1=t_2=t}$. Substitute t_1 for t in (184) and t_2 for t in (185). Then multiplication of (183) and (184) and use of the definition of $\hat{r}(t_1, t_2)$ yields

$$\frac{\partial^2 \hat{r}(t_1, t_2)}{\partial t_2 \partial t_1} + a\frac{\partial \hat{r}(t_1, t_2)}{\partial t_2} + b\frac{\partial \hat{r}(t_1, t_2)}{\partial t_1} + ab\hat{r}(t_1, t_2) = x(t_1)\, x(t_2). \tag{187}$$

In order to express (1?5) in terms of $\hat{y}(t_1, t_2)$ we must find an expression for $dy(t)/dt$ in terms of $\hat{y}(t_1, t_2)$. Since $\hat{y}(t, t) = y(t)$, the desired derivative will be the directional derivative of $\hat{y}(t_1, t_2)$ along the line $t_1 = t_2$, scaled by the factor $\sqrt{2}$ to obtain the proper rate of change. The directional derivative is given by the dot product of the gradient of $\hat{y}(t_1, t_2)$ with the unit vector in the direction of the 45° line. Hence we have the correspondence

$$\frac{dy(t)}{dt} \longleftrightarrow \sqrt{2}\,(\nabla \hat{y}(t_1, t_2)) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{\partial \hat{y}(t_1, t_2)}{\partial t_1} + \frac{\partial \hat{y}(t_1, t_2)}{\partial t_2}. \tag{188}$$

Repeating this operation, we find the correspondence for the second derivative.

$$\frac{d^2 y(t)}{dt^2} \longleftrightarrow \frac{\partial^2 \hat{y}(t_1, t_2)}{\partial t_1^2} + 2\,\frac{\partial^2 \hat{y}(t_1, t_2)}{\partial t_2 \partial t_1} + \frac{\partial^2 \hat{y}(t_1, t_2)}{\partial t_2^2}. \tag{189}$$

By using these results (186) can be extended to

$$\frac{\partial^2 \hat{y}}{\partial t_1^2} + 2\,\frac{\partial^2 \hat{y}}{\partial t_2 \partial t_1} + \frac{\partial^2 \hat{y}}{\partial t_2^2} + d\,\frac{\partial \hat{y}}{\partial t_1} + d\,\frac{\partial \hat{y}}{\partial t_2} + e\hat{y} = c\hat{r} + \frac{\partial \hat{r}}{\partial t_1} + \frac{\partial \hat{r}}{\partial t_2}. \tag{190}$$

We must now combine (190) and (187) to eliminate $\hat{r}(t_1, t_2)$. This may be accomplished as follows. Take the partial of both sides of (187) with respect to t_1 to obtain (191), and with respect to t_2 to obtain (192).

$$\frac{\partial^3 \hat{r}}{\partial t_2 \partial t_1^2} + a\,\frac{\partial^2 \hat{r}}{\partial t_2 \partial t_1} + b\,\frac{\partial^2 \hat{r}}{\partial t_1^2} + ab\,\frac{\partial \hat{r}}{\partial t_1} = \frac{dx(t_1)}{dt_1}\,x(t_2) \tag{191}$$

$$\frac{\partial^3 \hat{r}}{\partial t_2^2 \partial t_1} + a\,\frac{\partial^2 \hat{r}}{\partial t_2^2} + b\,\frac{\partial^2 \hat{r}}{\partial t_2 \partial t_1} + ab\,\frac{\partial \hat{r}}{\partial t_2} = x(t_1)\,\frac{dx(t_2)}{dt_2}. \tag{192}$$

Also, we take the partial of (190) with respect to t_1 to obtain (193), with respect to t_2 to obtain (194), and with respect to t_1 and t_2 to obtain (195).

$$\frac{\partial^3 \hat{y}}{\partial t_1^3} + 2\,\frac{\partial^3 \hat{y}}{\partial t_2 \partial t_1^2} + \frac{\partial^3 \hat{y}}{\partial t_2^2 \partial t_1} + d\,\frac{\partial^2 \hat{y}}{\partial t_1^2} + d\,\frac{\partial^2 \hat{y}}{\partial t_2 \partial t_1} + e\,\frac{\partial \hat{y}}{\partial t_1} = c\,\frac{\partial \hat{r}}{\partial t_1} + \frac{\partial^2 \hat{r}}{\partial t_1} + \frac{\partial^2 \hat{r}}{\partial t_2 \partial t_1} \tag{193}$$

$$\frac{\partial^3 \hat{y}}{\partial t_2 \partial t_1^2} + 2\,\frac{\partial^3 \hat{y}}{\partial t_2^2 \partial t_1} + \frac{\partial^3 \hat{y}}{\partial t_2^3} + d\,\frac{\partial^2 \hat{y}}{\partial t_2 \partial t_1} + d\,\frac{\partial^2 \hat{y}}{\partial t_2^2} + e\,\frac{\partial \hat{y}}{\partial t_2} = c\,\frac{\partial \hat{r}}{\partial t_2} + \frac{\partial^2 \hat{r}}{\partial t_2 \partial t_1} + \frac{\partial^2 \hat{r}}{\partial t_2^2} \tag{194}$$

$$\frac{\partial^4 \hat{y}}{\partial t_2 \partial t_1^3} + 2\,\frac{\partial^4 \hat{y}}{\partial t_2^2 \partial t_1^2} + \frac{\partial^4 \hat{y}}{\partial t_2^3 \partial t_1} + d\,\frac{\partial^3 \hat{y}}{\partial t_2 \partial t_1^2} + d\,\frac{\partial^3 \hat{y}}{\partial t_2^2 \partial t_1} + e\,\frac{\partial^2 \hat{y}}{\partial t_2 \partial t_1} = c\,\frac{\partial^2 \hat{r}}{\partial t_2 \partial t_1} + \frac{\partial^3 \hat{r}}{\partial t_2 \partial t_1^2} + \frac{\partial^3 \hat{r}}{\partial t_2^2 \partial t_1}. \tag{195}$$

Next, multiply through (187) by c, and add the resulting equation to (191) plus (192). Multiply through (194) by a, (193) by b, and (190) by ab, and add the sum of these new equations to (195). We then can write

$$cx(t_1)\, x(t_2) + x(t_1) \frac{dx(t_2)}{dt_2} + \frac{dx(t_1)}{dt_1} x(t_2) =$$

$$\frac{\partial^3 \hat{r}}{\partial t_2^2 \partial t_1} + \frac{\partial^3 \hat{r}}{\partial t_2 \partial t_1^2} + (a+b+c) \frac{\partial^2 \hat{r}}{\partial t_2 \partial t_1} + b \frac{\partial^2 \hat{r}}{\partial t_1^2} + a \frac{\partial^2 \hat{r}}{\partial t_2^2} + (ab+bc) \frac{\partial \hat{r}}{\partial t_1} + (ab+ac) \frac{\partial \hat{r}}{\partial t_2} + abc\hat{r}$$

$$= \frac{\partial^4 \hat{y}}{\partial t_2 \partial t_1^3} + 2 \frac{\partial^4 \hat{y}}{\partial t_2^2 \partial t_1^2} + \frac{\partial^4 \hat{y}}{\partial t_2^3 \partial t_1} + (a+2b+d) \frac{\partial^3 \hat{y}}{\partial t_2 \partial t_1^2} + (2a+b+d) \frac{\partial^3 \hat{y}}{\partial t_2^2 \partial t_1} + b \frac{\partial^3 \hat{y}}{\partial t_1^3}$$

$$+ a \frac{\partial^3 \hat{y}}{\partial t_2^3} + (ad+2ab+bd+e) \frac{\partial^2 \hat{y}}{\partial t_2 \partial t_1} + (ab+bd) \frac{\partial^2 \hat{y}}{\partial t_1^2} + (ab+ad) \frac{\partial^2 \hat{y}}{\partial t_2^2}$$

$$+ (abd+be) \frac{\partial \hat{y}}{\partial t_1} + (abd+ae) \frac{\partial \hat{y}}{\partial t_2} + abe\hat{y}$$

or

$$\frac{\partial^4 \hat{y}}{\partial t_2 \partial t_1^3} + 2 \frac{\partial^4 \hat{y}}{\partial t_2^2 \partial t_1^2} + \frac{\partial^4 \hat{y}}{\partial t_2^3 \partial t_1} + (a+2b+d) \frac{\partial^3 \hat{y}}{\partial t_2 \partial t_1^2} + (2a+b+d) \frac{\partial^3 \hat{y}}{\partial t_2^2 \partial t_1} + b \frac{\partial^3 \hat{y}}{\partial t_1^3}$$

$$+ a \frac{\partial^3 \hat{y}}{\partial t_2^3} + (ad+2ab+bd+e) \frac{\partial^2 \hat{y}}{\partial t_2 \partial t_1} + (ab+bd) \frac{\partial^2 \hat{y}}{\partial t_1^2} + (ab+ad) \frac{\partial^2 \hat{y}}{\partial t_2^2}$$

$$+ (abd+be) \frac{\partial \hat{y}}{\partial t_1} + (abd+ae) \frac{\partial \hat{y}}{\partial t_2} + abe\hat{y}$$

$$= cx(t_1)\, x(t_2) + x(t_1) \frac{dx(t_2)}{dt_2} + \frac{dx(t_1)}{dt_1} x(t_2). \tag{196}$$

We have thus obtained a single differential equation relating y(t) and x(t). The equation is a linear partial differential equation with constant coefficients.

This linear partial differential equation is particularly well suited to solution by means of the two-dimensional Laplace transform. Taking the transform of each side, we find

$$\Big[s_1^3 s_2 + 2s_1^2 s_2^2 + s_1 s_2^3 + (a+2b+d) s_1^2 s_2 + (2a+b+d) s_1 s_2^2 + bs_1^3 + as_2^3 + (ad+2ab+bd+e) s_1 s_2$$

$$+ (ab+bd) s_1^2 + (ab+ad) s_2^2 + (abd+be) s_1 + (abd+ae) s_2 + abe\Big] \hat{Y}(s_1, s_2)$$

$$= (s_1 + s_2 + c)\, X(s_1)\, X(s_2).$$

Factoring the polynomials in this expression and solving for $\hat{Y}(s_1, s_2)$, we have

$$\hat{Y}(s_1, s_2) = \frac{s_1 + s_2 + c}{(s_1+b)(s_2+a)\left[(s_1+s_2)^2+d(s_1+s_2)+e\right]} X(s_1)\, X(s_2). \tag{197}$$

We now note that

$$H_2(s_1, s_2) \triangleq \frac{s_1 + s_2 + c}{(s_1+b)(s_2+a)\left[(s_1+s_2)^2+d(s_1+s_2)+e\right]}$$

is the transform of the Volterra kernel, $h_2(\tau_1, \tau_2)$, of the system of Fig. 1 when the system is characterized by the integral equation

$$y(t) = \iint h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2. \tag{198}$$

In fact, taking the inverse transform of (197) we have

$$\hat{y}(t_1, t_2) = \iint h_2(\tau_1, \tau_2)\, x(t_1-\tau_1)\, x(t_2-\tau_2)\, d\tau_1 d\tau_2$$

from which (198) follows by setting $t_1 = t_2 = t$.

From this example the following observations may be made.

1. Given a system of equations that are the dynamic description of a nonlinear system, by extending the domain of definition from one dimension to two dimensions, we were able to find a single linear partial differential equation that also characterizes the system. That is, by extending from one dimension into two dimensions, a one-dimensional nonlinear problem was converted into a two-dimensional linear problem.

2. Equations 183-186 and Eq. 198 describe the same situation. A system that is characterized by a single integral equation is equivalently described by a set of several ordinary differential equations and a nondifferential equation. A description by one nonlinear ordinary differential equation does not seem to be possible.

3. Whenever a system is characterized by an n^{th}-degree Volterra kernel having a rational transform, a linear partial differential equation with constant coefficients can be found which relates the auxiliary output function $\hat{y}(t_1, \ldots, t_n)$ to the input function $x(t)$. If the kernel is of the class that can be realized exactly with a finite number of linear systems and multipliers, then an equivalent description by a set of ordinary differential equations and nondifferential equations can be found.

Although the example and observations presented here have not yet led to the solution of any problems that cannot be easily handled by other methods, it is felt that the viewpoint presented is unique and may lead to a deeper understanding of the properties of nonlinear systems.

IX. CONCLUSION

We have studied some techniques for the synthesis of nonlinear systems. The systems considered are those that can be characterized by a finite set of Volterra kernels: $\{h_n(\tau_1, \ldots, \tau_n): n = 0, 1, 2, \ldots, N\}$. The approach adopted throughout has been to consider the kernels one at a time, using as basic elements in the synthesis linear systems and multipliers.

We have presented a procedure for testing a given kernel transform to determine whether or not the kernel can be realized exactly with a finite number of linear systems and multipliers. The test is constructive. If it is possible to realize the kernel exactly, a realization is given by the test; if it is not possible to realize the complete kernel exactly, but it is possible to break the kernel up into several lower degree components, this will also be discovered by the test.

An extension to nonlinear systems of the impulse-train techniques of linear system theory is given. Although applicable in principle to higher degree systems, the use of impulse-train techniques as graphical methods is effectively limited to second-degree systems.

The use of digital systems is recognized as a powerful tool in modern system theory. We have developed properties of sampling in nonlinear systems, in order to facilitate the use of digital techniques in the synthesis of nonlinear systems. Bandlimiting in nonlinear systems is discussed, and delay line models for bandlimited systems are given.

The transform analysis of nonlinear sampled-data systems by means of the multidimensional z-transform is presented. Computation algorithms for input-output computations are given for direct computation from the multidimensional convolution sum, from the associated partial difference equation, and from a decomposition of the nonlinear sampled-data system into linear sampled-data systems.

A relationship between time-variant and time-invariant systems is presented, in which time-variant systems are shown to be related to time-invariant systems of higher degree. This enables one to use for linear time-variant systems the properties and techniques developed for second-degree time-invariant systems.

A note on the multidimensional formulation of nonlinear systems from the differential equation point of view is given; it is seen that some nonlinear problems in one dimension can be mapped into a linear problem in a higher dimensional space.

As with linear systems, the problem of the synthesis of a nonlinear system is the problem of finding a finite-dimensional state space in which the system may be described. One expects that an attack on the problem directly from the state space point of view may be fruitful.

APPENDIX A

Proofs of the properties of z-transforms given in Chapter V are given below.

A. 1 Proof of 5. a. 1

$$\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f[(k_1-b_1)T, \ldots, (k_n-b_n)T]\, z_1^{-k_1} \cdots z_n^{-k_n}$$

$$= z_1^{-b_1} \cdots z_n^{-b_n} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f[(k_1-b_1)T, \ldots, (k_n-b_n)T]\, z_1^{-(k_1-b_1)} \cdots z_n^{-(k_n-b_n)}$$

$$= z_1^{-b_1} \cdots z_n^{-b_n} F(z_1, \ldots, z_n)$$

A. 2 Proof of 5. a. 2

$$\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} e^{-a_1k_1T} \cdots e^{-a_nk_nT} f(k_1T, \ldots, k_nT)\, z_1^{-k_1} \cdots z_n^{-k_n}$$

$$= \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T, \ldots, k_nT) \left(e^{a_1T} z_1\right)^{-k_1} \cdots \left(e^{a_nT} z_n\right)^{-k_n}$$

$$= F\left(e^{a_1T} z_1, \ldots, e^{a_nT} z_n\right)$$

A. 3 Proof of 5. a. 3

$$-Tz_i \frac{\partial}{\partial z_i} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T, \ldots, k_nT)\, z_1^{-k_1} \cdots z_n^{-k_n}$$

$$= \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} (-T) f(k_1T, \ldots, k_nT)\, z_i \frac{\partial}{\partial z_i} \left[z_1^{-k_1} \cdots z_n^{-k_n}\right]$$

$$= \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} k_i T f(k_1T, \ldots, k_nT)\, z_1^{-k_1} \cdots z_n^{-k_n} \longleftrightarrow \tau_i f(\tau_1, \ldots, \tau_n)$$

.4 Proof of 5. a. 4

$$H(z_1,\ldots,z_n) = \sum_{p_n=-\infty}^{\infty} \cdots \sum_{p_1=-\infty}^{\infty} h_n(z_1,\ldots,z_n)\, z_1^{-p_1} \cdots z_n^{-p_n}$$

$$= \sum_{p_n=-\infty}^{\infty} \cdots \sum_{p_1=-\infty}^{\infty} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T,\ldots,k_nT)$$

$$\cdot\, g(p_1T-k_1T,\ldots,p_nT-k_nT)\, z_1^{-p_1} \cdots z_n^{-p_n}$$

$$= \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T,\ldots,k_nT) \sum_{p_n=-\infty}^{\infty} \cdots \sum_{p_1=-\infty}^{\infty}$$

$$\cdot\, g(p_1T-k_1T,\ldots,p_nT-k_nT)\, z_1^{-p_1} \cdots z_n^{-p_n}$$

$$= \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T,\ldots,k_nT)\, z_1^{-k_1} \cdots z_n^{-k_n}\, G(z_1,\ldots,z_n)$$

$$= F(z_1,\ldots,z_n)\, G(z_1,\ldots,z_n)$$

APPENDIX B

Proofs of the properties of modified z-transforms given in Section V, and the details of Example 6 are presented here.

B. 1 Proof of 5. b. 1

$$\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f[k_1T-\Delta_1T-(1-m_1)T, \ldots, k_nT-\Delta_nT-(1-m_n)T]\, z_1^{-k_1} \ldots z_n^{-k_n} =$$

$$z_1^{-1} \ldots z_n^{-1} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f\{k_1T-[1-(1+m_1-\Delta_1)]T, \ldots, k_nT-[1-(1+m_n-\Delta_n)]T\} z_1^{-k_1} \ldots z_n^{-k_n}$$

$$= z_1^{-1} \ldots z_n^{-1} F_m(z_1, 1+m_1-\Delta_1; \ldots ; z_n, 1+m_n-\Delta_n) \qquad \text{for } 0 \le m_i < \Delta_i < 1, \qquad i = 1, \ldots, n.$$

For $0 \le \Delta_i \le m_i < 1$, we may write the left side of the equation above as

$$\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f\{k_1T-[1-(m_1-\Delta_1]T, \ldots, k_nT-[1-(m_n-\Delta_n)]T\}\, z_1^{-k_1} \ldots z_n^{-k_n}$$

$$= F_m(z_1, m_1-\Delta_1; \ldots ; z_n, m_n-\Delta_n).$$

For shifts equal to an integral multiple of T, the proof of this property is the same as that given above for the ordinary z-transform.

B. 2 Proof of 5. b. 2

$$\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} e^{-a_1[k_1T-(1-m_1)T]} \ldots e^{-a_n[k_nT-(1-m_n)T]} f[k_1T-(1-m_1)T, \ldots, k_nT-(1-m_n)T]$$

$$\cdot\, z_1^{-k_1} \ldots z_n^{-k_n}$$

$$= e^{a_1(1-m_1)} \ldots e^{a_n(1-m_n)} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f[k_1T-(1-m_1)T, \ldots, k_nT-(1-m_n)T]$$

$$\cdot \left(e^{a_1T} z_1\right)^{-k_1} \ldots \left(e^{a_nT} z_n\right)^{-k_n}$$

$$= e^{a_1(1-m_1)} \ldots e^{a_n(1-m_n)} F_m\left(e^{a_1T} z_1, m_1; \ldots ; e^{a_nT} z_n, m_n\right).$$

B. 3 Proof of 5. b. 3

$$\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} [k_iT-(1-m_i)T]\, f[k_1T-(1-m_1)T, \ldots, k_nT-(1-m_n)T]\, z_1^{-k_1} \cdots z_n^{-k_n} =$$

$$T(m_i-1)\, F_m(z_1, m_1; \ldots; z_n, m_n) + \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} k_iTf[k_1T-(1-m_1)T, \ldots, k_nT-(1-m_n)T]$$

$$\cdot\, z_1^{-k_1} \cdots z_n^{-k_n}$$

$$= T[(m_i-1)\, F_m(z_1, m_1; \ldots; z_n, m_n) - z_i \frac{\partial}{\partial z_i} F_m(z_1, m_1; \ldots; z_n, m_n)].$$

B. 4 Proof of 5. b. 4

When $f(\tau_1, \ldots, \tau_n)$ is continuous from the right in each of the variables,

$$\lim_{\substack{m_i \to 0 \\ i=1,\ldots n}} F_m(z_1, m_1; \ldots; z_n, m_n) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T-T, \ldots, k_nT-T)\, z_1^{-k_1} \cdots z_n^{-k_n}$$

$$= z_1^{-1} \cdots z_n^{-1} F(z_1, \ldots, z_n).$$

When $f(\tau_1, \ldots, \tau_n)$ is continuous from the left in each of the variables,

$$\lim_{\substack{m_i \to 1 \\ i=1,\ldots n}} F_m(z_1, m_1; \ldots; z_n, m_n) = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} f(k_1T, \ldots, k_nT)\, z_1^{-k_1} \cdots z_n^{-k_n}$$

$$= F(z_1, \ldots, z_n).$$

B.5 Details of Example 6

Direct Transform:

$$F_1(z_1, z_2) = \sum_{k=0}^{\infty} \left(1 - e^{-2T(k+m^*-1)}\right)(z_1 z_2)^{-k}$$

$$+ \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \left(1 - e^{-2T(k_2+m^*-1)}\right) z_1^{-(k_1+k_2)} z_2^{-k_2}$$

$$+ \sum_{k_2=1}^{\infty} \sum_{k_1=0}^{\infty} \left(1 - e^{-2T(k_1+m^*-1)}\right) z_1^{-k_1} z_2^{-(k_2+k_1)}$$

$$= \frac{1}{1 - z_1^{-1} z_2^{-1}} - \frac{e^{-2T(m^*-1)}}{1 - e^{-2T} z_1^{-1} z_2^{-1}} - 1 + \frac{1}{1 - z_1^{-1}} + \frac{1}{1 - z_2^{-1}}$$

$$= \frac{1 - e^{-2T(m^*-1)} - z_1^{-1} z_2^{-1}\left(e^{-2T} - e^{-2T(m^*-1)}\right)}{\left(1 - e^{-2T} z_1^{-1} z_2^{-1}\right)\left(1 - z_1^{-1}\right)\left(1 - z_2^{-1}\right)}$$

Inverse Transform:

$$\frac{1}{1 - e^{-6T} z_1^{-1} z_2^{-1}} \frac{1}{1 - e^{-T} z_1^{-1}} \frac{1}{1 - e^{-3T} z_2^{-1}} = \left(1 + e^{-6T} z_1^{-1} z_2^{-1} + e^{-12T} z_1^{-2} z_2^{-2} + e^{-18T} z_1^{-3} z_2^{-3} + \ldots\right)$$

$$\cdot \left(1 + e^{-T} z_1^{-1} + e^{-2T} z_1^{-2} + e^{-3T} z_1^{-3} + \ldots\right)\left(1 + e^{-3T} z_2^{-1} + e^{-6T} z_2^{-2} + e^{-9T} z_2^{-3} + \ldots\right).$$

APPENDIX C

Proofs of properties of the transforms of causal functions in Section 5.3 are given low.

1 Proof of 5.c.1

For causal $f(\tau_1, \ldots, \tau_n)$ we have

$$F(z_1, \ldots, z_n) = \sum_{k_n=0}^{\infty} \cdots \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} f(k_1 T, k_2 T, \ldots k_n T)\, z_1^{-k_1} z_2^{-k_2} \ldots z_n^{-k_n}.$$

ien

$$\lim_{z_1 \to \infty} F(z_1, \ldots, z_n) = \sum_{k_n=0}^{\infty} \cdots \sum_{k_2=0}^{\infty} f(0, k_2 T, \ldots, k_n T)\, z_2^{-k_2} \ldots z_n^{-k_n}.$$

For $z_2, \ldots, z_n$ a similar relation holds. Hence

$$\lim_{z_i \to \infty} F(z_1, \ldots, z_n) \longleftrightarrow \lim_{\tau_i \to 0} f(\tau_1, \ldots, \tau_n).$$

2 Proof of 5.c.2

$_1 + T, \tau_2, \ldots, \tau_n) - f(\tau_1, \ldots, \tau_n) \longleftrightarrow$

$$\lim_{p \to \infty} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-p}^{p} [f(k_1 T+T, k_2 T, \ldots, k_n T) - f(k_1 T, \ldots, k_n T)]\, z_1^{-k_1} \ldots z_n^{-k_n}.$$

ıe expression on the right may be written

$$\sum_{=-\infty}^{\infty} \cdots \sum_{k_2=-\infty}^{\infty} \lim_{p \to \infty} \sum_{k_1=-p}^{+p} [f(k_1 T+T, k_2 T, \ldots, k_n T) - f(k_1 T, \ldots, k_n T)]\, z_1^{-k_1} z_2^{-k_2} \ldots z_n^{-k_n}.$$

For $z_1 = 1$, the inner sum is

$$\lim_{p\to\infty} \sum_{k_1=-p}^{0} [f(k_1T+T, k_2T, \ldots, k_nT) - f(k_1T, \ldots, k_nT)]$$

$$+ \sum_{k_1=0}^{+p} [f(k_1T+T, k_2T, \ldots, k_nT) - f(k_1T, \ldots, k_nT)]$$

$$- f(T, k_2T, \ldots, k_nT) + f(0, k_2T, \ldots, k_nT).$$

These sums telescope, to give

$$\lim_{p\to\infty} f[(p+1)T, k_2T, \ldots, k_nT] - f(0, k_2T, \ldots, k_nT) + f[(1-p)T, k_2T, \ldots, k_nT]$$

$$+ f(T, k_2T, \ldots, k_nT) - f(T, k_2T, \ldots, k_nT) + f(0, k_2T, \ldots, k_nT).$$

Now since $f(\tau_1, \ldots, \tau_n)$ is causal by hypothesis, this becomes

$$\lim_{p\to\infty} f[(p+1)T, k_2T, \ldots, k_nT].$$

Hence, if this limit exists,

$$\lim_{\tau_1\to\infty} f(\tau_1, \ldots, \tau_n) \longleftrightarrow \lim_{z_1\to 1} (z_1-1)\, F(z_1, \ldots, z_n).$$

A similar relation holds for $i = 2, \ldots, n$.

C.3 Proof of Eqs. (84) and (85)

We prove only Eq. 85 from which (84) follows also by 5.b.4. Define $\delta_T(\tau_1, \ldots, \tau_n)$ by

$$\delta_T(\tau_1, \ldots, \tau_n) = \left[\sum_{k_1=0}^{\infty} u_o(\tau_1 - k_1T)\right] \cdots \left[\sum_{k_n=0}^{\infty} u_o(\tau_n - k_nT)\right].$$

Then the Laplace transform of $\delta_T(\tau_1, \ldots, \tau_n)$ is

$$\Delta_T(s_1, \ldots, s_n) = \prod_{i=1}^{n} \frac{1}{1 - e^{-s_iT}}.$$

For causal $f(\tau_1, \ldots, \tau_n)$ we may write the modified z-transform $F_m(z_1, m_1; \ldots; z_n, m_n)$ as the z-transform of

$$f^*(\tau_1, m_1; \ldots; \tau_n, m_n) = f(\tau_1 - T + m_1T, \ldots, \tau_n - T + m_nT)\ \delta_T(\tau_1, \ldots, \tau_n)$$

$$= f(\tau_1 + m_1T - T, \ldots, \tau_n + m_nT - T)\ \delta_T(\tau_1 - T, \ldots, \tau_n - T).$$

t is clear that $F^*(s_1,\dots,s_n)$, the Laplace transform of $f^*(\tau_1,\dots,\tau_n)$, evaluated at $e^{s_iT} = z_i$, $i = 1,\dots,n$ is the z-transform of $f^*(\tau_1, m_1;\dots;\tau_n, m_n)$. The Laplace transorm of a product of functions results in the complex convolution of the Laplace transorms of the factors. Now the Laplace transform of $f(\tau_1+m_1T,\dots,\tau_n+m_nT)$ is

$$e^{s_1m_1T}\cdots e^{s_nm_nT}\,F(s_1,\dots,s_n),$$

where $F(s_1,\dots,s_n)$ is the Laplace transform of $f(\tau_1,\dots,\tau_n)$. We then have

$$F_m(z_1,m_1;\dots;z_n,m_n)\Big|_{\substack{z_i=e^{s_iT}\\ i=1,\dots,n}} = e^{-s_1T}\cdots e^{-s_nT}\,e^{s_1m_1T}\cdots e^{s_nm_nT}\,F(s_1,\dots,s_n)$$

$$\circledast\ \Delta_T(s_1,\dots,s_n)$$

where $\circledast$ denotes multidimensional complex convolution. This is the expression given xplicitly in (85). We note that in this expression, because of the nature of the Laplace nversion integral,[24] it is assumed that $f(\tau_1,\dots,\tau_n)$ is defined as the average value at ump discontinuities.

APPENDIX D

Equations stated in Section V without proof are justified below.

D. 1 Proof of Eqs. 116 and 117

We shall prove (116) for $n = 2$. The extension to the higher dimensional case is clear.

$$y_{(2)}(kT, kT) = \left(\frac{1}{2\pi j}\right)^2 \oint_{\Gamma_1} \oint_{\Gamma_2} z_1^{k-1} z_2^{k-1} Y_{(2)}(z_1, z_2)\, dz_1 dz_2.$$

Let $z = z_1 z_2$; then $dz = z_1\, dz_2$ and

$$y_{(2)}(kT, kT) = \left(\frac{1}{2\pi j}\right)^2 \oint_{\Gamma} \oint_{\Gamma} z^{k-1} z_1^{-1} Y_{(2)}\left(z_1, \frac{z}{z_1}\right) dz_1 dz;$$

and hence

$$Y(z) = \frac{1}{2\pi j} \oint_{\Gamma} z_1^{-1} Y_{(2)}\left(z_1, \frac{z}{z_1}\right) dz_1.$$

For the modified z-transform (117), we have

$$y_{(2)}[(k_1-1+m_1)T, (k_2-1+m_2)T] = \left(\frac{1}{2\pi j}\right)^2 \oint_{\Gamma} \oint_{\Gamma} z_1^{k_1-1} z_2^{k_2-1} Y_{(2)m}(z_1, m_1; z_2, m_2)\, dz_1 dz_2.$$

Setting $k_1 = k_2 = k$ and $m_1 = m_2 = m$ yields

$$y[(k-m+1)T] = \left(\frac{1}{2\pi j}\right)^2 \oint_{\Gamma} \oint_{\Gamma} (z_1 z_2)^{k-1} Y_{(2)m}(z_1 m; z_2, m)\, dz_1 dz_2.$$

From this point the proof follows that above for the z-transform, and (118) follows for $n = 2$. The extension to higher dimensions is clear.

D. 2 Derivation of Eq. 118

$$\frac{1}{2\pi j} \oint_{\Gamma} z_1^{-1} A(z) \frac{1}{1\ e^{-aT} z_1^{-1}} \frac{1}{1 - e^{-bT}\left(\frac{z}{z_1}\right)^{-1}}\, dz_1 =$$

$$A(z) \frac{1}{2\pi j} \oint_{\Gamma} \frac{1}{z_1 - e^{-aT}} \frac{e^{bT} z}{e^{bT} z - z_1}\, dz_1 = A(z) \frac{1}{1 - e^{-(a+b)T} z^{-1}}.$$

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